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Finite Element Solution of the Helmholtz Equation with High Wave Number Part I: The h-Version of the FEM

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Abstract—The paper addresses the properties of finite element solutions for the Helmholtz equation. The h-version of the finite element method with piecewise linear approximation is applied to a one-dimensional model problem. New results are shown on stability and error estimation of the discrete model. In all propositions, assumptions are made on the magnitude of hk only, where k is the wavelength and h is the stepwidth of the FE-mesh. Previous analytical results had been shown with the assumption that k^2h is small. For medium and high wavenumber, these results do not cover the meshsizes that are applied in practical applications. The main estimate reveals that the error in H^1-norm of discrete solutions for the Helmholtz equation is polluted when k^2h is not small. The error is then not quasioptimal; i.e., the relation of the FE-error to the error of best approximation generally depends on the wavenumber k. It is noted that the pollution term in the relative error is of the same order as the phase lead of the numerical solution. In the result of this analysis, thorough and rigorous understanding of error behavior throughout the range of convergence is gained. Numerical results are presented that show sharpness of the error estimates and highlight some phenomena of the discrete solution behavior. The h-p-version of the FEM is studied in Part II.

Keywords—Helmholtz equation, Finite element method, Elliptic, Partial differential equation.

1. INTRODUCTION

Boundary value problems for the Helmholtz equation

$$\Delta u + k^2 u = f,$$

where k is the wave number, arise in a number of physical applications [1], in particular in problems of wave scattering and fluid-solid-interaction [2].

The quality of discrete numerical solutions to the Helmholtz equation depends significantly on the physical parameter k. It is clear and well known that the stepwidth h of meshes for finite element or finite difference computations should be adjusted to the wavenumber k. In practice, one usually follows a "rule of the thumb" of the form [3, p. 71]

$$kh = \text{ const.}$$

In computations with low wavenumber, this rule leads to sufficiently correct results. The quality of numerical results, however, deteriorates if the wavenumber k increases. Thus, Bayliss *et al.* [4]

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX solve the two-dimensional Helmholtz equation by piecewise linear FEM and tabulate the errors in L^2-norm. The results show that the errors grow with k for kh = const. On the other hand, the errors are bounded on a series of meshes with $k^3h^2 \approx$ const. A convergence theorem is stated in [4] *under the assumption that k^2h is sufficiently small.* As a consequence of this theorem, it is shown that for certain classes of data the relative errors are $O((kh)^p)$ in H^1-norm and $O(k(kh)^{p+1})$ in L^2-norm, where p is the order of polynomial approximation. The theorem from [4] has been rigorously proven for one-dimensional Helmholtz problems and piecewise linear approximation by Aziz *et al.* [5] and Douglas *et al.* [6]. In particular, it is shown that, if k^2h is sufficiently small, the error in H^1-seminorm satisfies a quasioptimal estimate

$$|u - u_{fe}|_1 \leq C \inf_{v \in V_h} |u - v|_1,$$

where V_h is the finite element subspace and C is a constant that does not depend on k and h.

However, the assumption on k^2h is unsatisfactory from a practical point of view since it generally holds on very fine mesh only.

To the knowledge of the authors, no error estimates for finite element solutions of the Helmholtz equation are known in the practically relevant case when *the magnitude of kh is constrained.* In this paper, we show for a one-dimensional model problem new results on stability and error estimation that hold under assumptions on the magnitude of kh only. This is called the *preasymptotic* case, whereas statements with the assumption that k^2h is small are called *asymptotic.* The paper is the first in a series dealing with the Galerkin finite element method for Helmholtz problems. In this first part, we restrict ourselves to piecewise linear approximation ($p = 1$) and concentrate on the H^1-norm of the error.

As a result of this analysis, a thorough and rigorous understanding of the error behavior of the finite element solution *throughout the range of convergence* is gained for the most simple case. However, as investigation of a fluid-solid interaction problem [7] and of the two-dimensional Helmholtz equation [8,9] have shown, the results of this basic investigation are well suited to explain the error behavior of more complicated Helmholtz problems.

In particular, it will be shown here that the relative error of the FE-solution in H^1-seminorm generally can be written as

$$\tilde{e}_1 \leq C_1 kh + C_2 k^3 h^2.$$

The first member on the right hand side reflects the approximation error which is of local character; it is present also in the asymptotic estimates. The second part is due to numerical pollution. This is a global effect that can be connected to a phase lead of the numerical solution. Note that the asymptotic estimate

$$\tilde{e}_1 \leq Ckh$$

follows from the preasymptotic estimate if k^2h is small. The effect of numerical pollution in the H^1-estimates is asymptotically negligible which leads to the previously known estimates.

As a prerequisite to the error estimate, the Babuška-Brezzi constant is computed here, both on the full space and in the finite element subspace. The constant is found to be of order k^{-1} in both cases. This is in correlation with previous numerical results reported by Demkowicz [10, p. 83] for a one-dimensional acoustic fluid-structure interaction problem.

We remark that the observation of the phase lead in discrete solutions for Helmholtz problems has given rise to specific modifications of the finite element method (e.g., the Galerkin Least Squares (GLS) method [3,11,12]. These methods can be interpreted in a broader sense as generalized finite element methods [9,13]. The reduction of the phase lead achieved by the GLS-method is equivalent to raising the order of k, h in the pollution term of the preasymptotic error [13]. In one dimension, the phase error can be eliminated without sacrificing the optimal order of convergence. In two dimensions, it is not possible to eliminate pollution entirely by *any* modification of the Galerkin finite element approach [12,13]. A generalized FEM that leads to minimal phase error for arbitrary wave direction in two dimensions is presented in [9].

The paper is organized as follows. We start (Section 2) with a recollection of existence, uniqueness and stability results in the strong sense. We then show existence-uniqueness for the weak solution and compute the Babuška-Brezzi constant. These results are the prerequisite for the main subject, the analysis of the finite element solution (Section 3). We first (Section 3.1) recall a standard approximation result in H^1 showing that the relative approximation error is $O(hk)$. We then (Section 3.2) formulate and prove a statement of existence-uniqueness for the finite element solution following Douglas *et al.* [6]. The proof is outlined in detail in order to keep track of all restrictions on h and k. The essence of the argument is that the finite element solution is quasioptimal provided the magnitude of hk^2 is sufficiently small. We then turn to the preasymptotic analysis where we make assumptions on the magnitude of hk only. Here, the finite element solution is analyzed via it's Green's function representation.[1] We investigate stability and show that on the finite-dimensional level the B-B-constant is of order k^{-1}. We then show that the relative error in H^1-norm is bounded if hk and h^2k^3 are appropriately constrained. In the numerical evaluation (Section 4), we present results from various computational experiments, applying and illustrating the main results of our study. We show, in particular, that the restriction of hk^2 is indeed necessary for quasioptimality of the finite element solution. The numerical experiments also prove that the theoretical error estimates are sharp.

2. THE MODEL PROBLEM

In this section, we prove existence-uniqueness of the solution to the one-dimensional reduced wave equation with Dirichlet and nonreflecting boundary conditions. We analyze the cases $u \in H^2(0,1)$ and $u \in H^1(0,1)$ separately and show that different stability conditions apply for these two cases. The construction of the Green's function to the problem is essential to both proofs.

2.1. The Boundary Value Problem

Let $\Omega = (0,1)$ and let on $\overline{\Omega}$ the boundary value problem $Lu = -f$ be given:

$$u''(x) + k^2u(x) = -f(x), \tag{2.1}$$

$$u(0) = 0, \tag{2.2}$$

$$u'(1) - iku(1) = 0, \tag{2.3}$$

where, for simplicity, $f(x) \in C^1(0,1)$ and $k \equiv \text{const.}, k \in \mathbb{R}, k > 0$.

Physically, if u is the variation of pressure in an acoustic medium at a fixed time, equation (2.1) is the equation of a plane wave with (nondimensional) wave number

$$k = \frac{\omega L}{c},$$

where ω is a given frequency, L is the measure of the domain and c is the speed of sound in the acoustic medium. In $x = 0$, a Dirichlet boundary condition is given (prescribed pressure); the mixed boundary condition in $x = 1$ is a Robin condition which in the one-dimensional case is equivalent to the Sommerfeldt radiation condition.

Notation

By $L^2(\Omega) := H^o(\Omega)$, we denote the space of all square-integrable complex-valued functions equipped with the inner product

$$(v, w) := \int_\Omega v(x)\bar{w}(x)\,dx$$

[1]The analysis is thus limited to uniform meshes. However, similar error behavior has been observed in numerical computation on highly irregular meshes [13].

and the norm

$$||w|| := \sqrt{(w,w)}.$$

We use the notation $H^s(\Omega)$ for the the Sobolev spaces of (integer) order s in the usual sense. Besides the usual full norm on H^s, we will also consider the seminorm

$$|u|_s = ||\partial^s u||,$$

where $\partial^s u$ is the s-derivative of u in the distributional sense. Note that for functions satisfying a Dirichlet condition (2.2), the seminorm $|u|_1$ is equivalent to the full H^1-norm $\|u\|_1 = (|u|_1^2 + \|u\|^2)^{1/2}]$.

Existence and Uniqueness in $H^2(0,1)$

The BVP (2.1)–(2.3) has a unique solution in the space $H^2(0,1)$. For the proof see, e.g., [5]. The existence of the solution is concluded from the following construction.

Inverse Operator

The Green's function of the BVP (2.1)–(2.3) is

$$G(x,s) = \frac{1}{k}\begin{cases} \sin kx e^{iks}; & 0 \le x \le s, \\ \sin ks e^{ikx}; & s \le x \le 1. \end{cases} \tag{2.4}$$

The solution $u(x)$ of (2.1)–(2.3) exists for all $k > 0$ and can be written as

$$u(x) = \int_0^1 G(x,s) f(s)\, ds.$$

LEMMA 1. *Let $u \in H^2(0,1)$ be the solution to the BVP (2.1)–(2.3). Then, if $f \in L^2(0,1)$*

$$||u|| \le k^{-1}||f||, \tag{2.5}$$

$$|u|_1 \le ||f||, \tag{2.6}$$

$$|u|_2 \le (1+k)||f||. \tag{2.7}$$

PROOF. See Douglas *et al.* [6]. ∎

REMARK 1. The aforementioned results are valid also for the adjoint problem (2.1), (2.2) and

$$u'(1) + iku(1) = 0.$$

2.2. Variational Formulation and Weak Solution

Consider the variational problem. Find $u \in V_1$ such that

$$\mathcal{B}(u,v) = \int_0^1 \left(u'(x)\bar{v}'(x) - k^2 u(x)\bar{v}(x)\right) dx - iku(1)\bar{v}(1) = \mathcal{F}(v), \tag{2.8}$$

where

$$\mathcal{F}(v) = \int_0^1 f(x)\bar{v}(x)\, dx, \tag{2.9}$$

holds for all $v \in V_2$. With

$$V_1 = V_2 = H^1_{(o)}(0,1) := \left\{ v \in H^1(0,1) \wedge v(0) = 0 \right\}, \tag{2.10}$$

this problem (2.8) is equivalent to the BVP (2.1)–(2.3) in the sense that for sufficiently smooth data any weak solution of (2.8) is a "strong" solution of (2.1)–(2.3).

For test functions $v \in H^1(0,1)$, the problem (2.8) is well defined if the data f lies at least in the dual space

$$H^{-1}(0,1) := \left\{ f \;\middle|\; |f|_{-1} := \sup_{v \in H^1_{(o)}(\Omega)} \frac{|\int_0^1 fv|}{|v|_1} < \infty \right\}.$$

Continuity of the form $\mathcal{B}$

Applying Poincaré's inequality, we obtain the continuity estimate

$$|\mathcal{B}(u,v)| \le C_o(k)|u|_1|v|_1,$$

with $C_o = 1 + k + k^2$.

Existence-uniqueness of the weak solution

We first show uniqueness. It suffices to show that $u = 0$ is the only homogeneous solution of (2.8). Hence, equation (2.8) hold with $\mathcal{F}(v) = 0$ for all v. Then for $v = u$,

$$\mathcal{B}(u,u) = \int_0^1 \left(u'(x)\bar{u}'(x) - k^2 u(x)\bar{u}(x)\right) dx - iku(1)\bar{u}(1) = 0.$$

Since the right-hand side of this equation is real, it follows that $u(1) = 0$, hence,

$$\forall v \in V: \quad \int_0^1 u'\bar{v}'dx = k^2 \int_0^1 u\bar{v}\,dx.$$

Taking $v = x$, we have

$$0 = u(1) - u(0) = \int_0^1 u'dx = k^2 \int_0^1 ux\,dx.$$

Assume now $\int_0^1 ux^n dx = 0$ for some natural n, then partial integration yields

$$0 = -\frac{1}{n+1}\int_0^1 u'x^{n+1}\,dx = \frac{k^2}{(n+1)(n+2)}\int_0^1 ux^{n+2}\,dx.$$

It follows by induction that

$$0 = \int_0^1 ux^s\,dx, \qquad s = 1,3,5,\ldots.$$

Since, as a consequence from Müntz's theorem [14, p. 45], the set

$$\text{span }\{x^s \mid s = 1,3,5,\ldots\}$$

is dense in $L^2(0,1)$, we conclude that $u \equiv 0$.

For the proof of existence, we observe that for the form $\mathcal{B}$, a Gårdings inequality

$$\operatorname{Re}(\mathcal{B}(u,u)) + C\|u\|^2 \ge \|u\|_1^2 \tag{2.11}$$

holds for $C = C(k) = 1 + k^2$. We then have (see, e.g., [15, p. 194]) the alternative statement: either there exists a nontrivial solution of the homogeneous problem $Lu = 0$ with Dirichlet data 0, or a solution of $Lu = f$ with Dirichlet data 0 exists for every sufficiently regular f. Since uniqueness has been proved, existence follows. The proof is completed.

REMARK 2. As in the strong case, we remark that existence-uniqueness holds obviously also for the adjoint form

$$\mathcal{B}^*(u,v) = \int_0^1 \left(u'(x)\bar{v}'(x) - k^2 u(x)\bar{v}(x)\right) dx + iku(1)\bar{v}(1).$$

Stability in H^1-norm and Babuška-Brezzi-constant

Stability in the weak case $f \in H^{-1}(\Omega)$ is concluded from the following theorem.

THEOREM 1. *Let $V = H^1_{(o)}(0,1)$ and $\mathcal{B} : V \times V \to \mathbb{C}$ as defined in equation (2.8). The Babuška-Brezzi stability constant*

$$\gamma := \inf_{u \in V} \sup_{v \in V} \frac{|\mathcal{B}(u,v)|}{|u|_1 |v|_1}$$

is of order k^{-1}; more precisely, there exist positive constants C_1, C_2 not depending on k such that

$$\frac{C_1}{k} \leq \gamma \leq \frac{C_2}{k}. \tag{2.12}$$

PROOF. Let us first proof the left inequality of (2.12). We will show that for any given $u \in V$ there exists an element $v \in V$ such that

$$|\mathcal{B}(u,v)| \geq \frac{C}{k} |u|_1 \, |v|_1. \tag{2.13}$$

Let $u \in V$ be given. Define $v := u + z$ where z is a solution of the problem

$$\forall w \in V : \quad \mathcal{B}(w,z) = k^2 (w,u). \tag{2.14}$$

The solution z exists and is uniquely defined. Furthermore, since $u \in H^1(0,1)$, z is a solution of the BVP (2.1)–(2.3) with data $k^2 u$, hence $z = k^2 \int_0^1 G(x,s) u(s)$ with the Greens function $G(x,s)$ from equation (2.4). Then

$$\begin{aligned} |\mathcal{B}(u,v)| &\geq \operatorname{Re} \mathcal{B}(u,v) \\ &= \operatorname{Re} \left(\mathcal{B}(u,u) + \mathcal{B}(u,z)\right) \\ &= \operatorname{Re} \left(\mathcal{B}(u,u) + \mathcal{B}(u,z) + k^2(u,u) - k^2(u,u)\right) \\ &= \operatorname{Re} \mathcal{B}(u,u) + k^2 \|u\|^2 = |u|_1^2. \end{aligned}$$

Now, if we show that

$$|u|_1 \geq \frac{C}{k} |v|_1, \tag{2.15}$$

we have proved inequality (2.13) and the inf-sup-condition follows.

To obtain inequality (2.15), integrate by parts the Green's function representation of z,

$$z(x) = k^2 \left(H(x,1) u(1) - \int_0^1 H(x,s) u'(s) \, ds \right), \tag{2.16}$$

where

$$H(x,s) := \int_0^s G(x,t) \, dt.$$

Differentiating this equation and taking absolute values, we get by triangular inequality

$$\begin{aligned} |z'(x)| &\leq k^2 \left(|H_x(x,1)| \, |u(1)| + \int_0^1 |H_x(x,s) u'(s)| \, ds \right) \\ &\leq k^2 \left(|H_x(x,1)| + \|H_x\| \right) |u|_1 \end{aligned}$$

By direct computation, $|H_x(x,1)| \leq 1/k$, $\|H_x\| \leq 1/k$, hence,

$$|z|_1 \leq 2k |u|_1.$$

Consequently,

$$|v|_1 \le |u|_1 + |z|_1 \le (1+2k)|u|_1$$

and, finally,

$$|u|_1 \ge \frac{C}{k}|v|_1$$

for $k > 1$. Together with equation (2.15), this validates the upper bound of the B-B-constant.

To prove the lower bound, it is sufficient to find some function $z_o(x) \in V$ for which

$$\forall v: \ \frac{|\mathcal{B}(z_o, v)|}{|z_o|_1} \le \frac{C}{k}|v|_1.$$

Consider the function

$$z_o(x) = \varphi(x)\,\frac{\sin kx}{k}$$

where $\varphi \in C^\infty(0,1)$ does not depend on k and is chosen such that

$$z_o(0) = z_o(1) = z_o'(0) = z_o'(1) = 0. \tag{2.17}$$

We further require

$$|z_o|_1 \ge \alpha$$

for some $\alpha > 0$, not depending on k (take, e.g., $\varphi(x) = x(x-1)^2$). Then

$$\forall v \in V: \ \ \frac{|\mathcal{B}(z_o, v)|}{|z_o|_1} \le \frac{1}{\alpha}|\mathcal{B}(z_o, v)|$$

and with equations (2.17), we obtain by partial integration

$$\forall v \in V: \ \ \mathcal{B}(z_o, v) = -\int_0^1 (z_o'' + k^2 z_o)\bar{v}.$$

Direct computation shows that

$$z_o'' + k^2 z_o = \varphi''\,\frac{\sin kx}{k} + 2\varphi'(x)\cos kx.$$

Define

$$u(x) := \int_0^x \left(z_o''(s) + k^2 z_o(s)\right) ds, \tag{2.18}$$

then

$$|\mathcal{B}(z_o, v)| = \left|u(1)\bar{v}(1) - \int_0^1 u(x)\bar{v}'(x)\,dx\right| \le (|u(1)| + \|u\|)|v|_1.$$

On the other hand, integrating zy parts in equation (2.18), it is easy to see that

$$|u(1)| \le \frac{1}{k}\|\varphi''\|_\infty$$

and

$$\|u\| \le \frac{1}{k}\left(\|\varphi''\|_\infty + 2\|\varphi'\|_\infty\right).$$

Hence, there exists a constant C such that

$$(|u(1)| + \|u\|) \le \frac{C}{k}.$$

Consequently,

$$\forall v \in V: \ \ |\mathcal{B}(z_o, v)| \le \frac{C}{k}\,|v|_1$$

and the proof is completed. ∎

From general theory [16, p. 112] we then have the following corollary.

COROLLARY 1. *Let $u \in H^1(0,1)$ be a solution of the variational problem (2.8). Then the stability estimate*

$$|u|_1 \le Ck|f|_{-1}$$

holds for constant C not depending on k.

3. FINITE ELEMENT SOLUTION

Following preliminary definitions, we state approximability of the exact solution as a direct conclusion from the approximation properties of the finite element space and stability (3.1). We then study the conditions for discrete stability and quasioptimal error estimates in the asymptotic range.

After that, we proceed to the study of the finite element solution in the preasymptotic range (3.2). We show the inf-sup condition and prove the main theorem, stating an error estimate in H^1-norm with assumptions on the magnitude of hk only. The section is concluded with some comments.

3.1. Approximability and Quasioptimal Error Estimate

Notation

Let on Ω a uniform mesh of $n+1$ nodes

$$X_h = \left\{ x_j = \frac{j}{n},\ j = 0, 1, \ldots, n \right\} \subset [0,1] \tag{3.1}$$

be given. The stepsize is $h = 1/n$. The intervals $[x_{j-1}, x_j]$ are called finite elements. We define the subspace $S_h(\Omega) \subset H^1(\Omega)$ as the set of all functions $u \in H^1(\Omega)$ such that the restriction of u to any element $[x_{i-1}, x_i]$ is a linear function. We further define the subspace

$$V_h = S_h[0,1) := \{v \in S_h(0,1), v(0) = 0\}.$$

A function $u \in V_h$ is called the finite element solution of the variational problem (2.8) if $\mathcal{B}(u, v) = \mathcal{F}(v)$ for all $v \in V_h$.

Further, a function defined on X_h is called a *mesh function* and will be referred to by subscript h. For a mesh function $u = u_h$, we will denote left and right differences, respectively, by

$$d^i u := \frac{u(x_i) - u(x_{i-1})}{h_i}; \qquad D^i u := \frac{u(x_{i+1}) - u(x_i)}{h_{i+1}}.$$

In the linear space of mesh functions, an inner product in L^2-analogy is defined by

$$(f_h, g_h)_h = h \sum_{j=1}^{n} f_j \bar{g}_j.$$

We will denote the discrete L^2-norm by $\|\cdot\|$. The discrete analogon to the H^1-seminorm is given by

$$|u_h|_1^2 = h \sum_{i=1}^{n} \left| d^i u_h \right|^2.$$

Note that for any piecewise linear function u with nodal points on X_h, we have $|u|_1 = |u_h|_1$, i.e., the discrete and exact H^1-norms are identical. We will use the discrete Dirac symbol defined as

$$\delta_{ij} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Approximation properties of $S_h(\Omega)$

It is well known that in one dimension, the best piecewise linear approximation in H^1-seminorm to a function $u \in H^1(\Omega)$ is the interpolant u_I. Furthermore, if $u \in H^2(\Omega)$, there holds the following lemma.

LEMMA 2. *Let $u \in H^2(0,1)$ and $u_I \in S_h(0,1)$ be the piecewise linear interpolant of u. Then*

$$\inf_{v\in S_h} \|u-v\| \le \|u-u_I\| \le \left(\frac{h}{\pi}\right)^2 |u|_2, \tag{3.2}$$

$$\inf_{v\in S_h} |u-v|_1 = |u-u_I|_1 \le \left(\frac{h}{\pi}\right) |u|_2, \tag{3.3}$$

$$\|u-u_I\| \le \left(\frac{h}{\pi}\right) |u-u_I|_1. \tag{3.4}$$

PROOF. See, e.g., [17, p. 45]. ∎

H^1-approximability for Helmholtz problems now immediately follows.

THEOREM 2. *Let $u \in H^2(\Omega)$ be the solution of the variational problem (2.8)—or, equivalently, of the BVP (2.1)–(2.3)—for given data $f \in L^2(\Omega)$. Then*

$$|u-u_I|_1 \le \frac{h}{\pi}(1+k)\|f\|.$$

PROOF. Combine Lemmas 1 and 2. ∎

We now reproduce a quasioptimal error estimate shown by Douglas *et al.* [6], paying special attention to the constants involved in the estimates. The proof is detailed in the Appendix.

THEOREM 3. *Let $f \in L^2(0,1)$ and let $u \in H^2(0,1)$ be the exact and $u_{fe} \in S_h[0,1)$ be the finite element solutions of the BVP (2.1)–(2.3), respectively. Assume that h and k are such that the denominators of the constants in the following estimates are positive.*

Then

$$|u-u_{fe}|_1 \le C_s \inf_{v\in V_h} |u-v|_1 \tag{3.5}$$

holds with

$$C_s := \frac{2\left(1+\left(\frac{hk}{2\pi}\right)^2\right)^{1/2}}{\left(\frac{1}{2}-6C_1^2k^2h^2(1+k)^2\right)^{1/2}}$$

and

$$C_1 := \frac{2}{\left(1-2(1+k)\frac{k^2h^2}{\pi^2}\right)\pi}.$$

Furthermore,

$$|u-u_{fe}|_1 \le C_s\,(1+k)\,h\,\|f\|. \tag{3.6}$$

PROOF. See Appendix A. ∎

Note that, for the denominator of C_s to be positive, the magnitudes of $(hk)^2$, h^2k^3 and h^2k^4 must be small. The term $(hk/2\pi)^2$ in the numerator can then be omitted. Hence if h and k fulfill the assumptions of the theorem, the finite element solution converges like the best approximation. It will be shown by numerical experiment that a bound on k^2h is also necessary for this quasioptimal behavior.

It is another question whether the assumptions of the theorem are necessary to bound the finite element error by some finite magnitude (like, e.g., an *a priori* given tolerance). The following simple computation indicates that this is not the case for high k. Let $hk^2 \le \alpha$ for some $\alpha > 0$. Then $h \le \alpha/k^2$ and

$$|u-u_{fe}|_1 \le C_2(1+k)\frac{1}{k^2}\|f\|;$$

hence, the error estimates of the theorem *tend towards 0* (while they have only to be bounded for practical purposes) as k is increased.

3.2. Stability and Error Estimation in the Preasymptotic Range

Global FE-equations and discrete fundamental system

After assembling the local equations (2.8) and multiplying the whole set by h, we arrive at a set of linear equations for the mesh-function $u_h = u_{fe}|_{X_h}$:

$$L_h u_h = r_h. \tag{3.7}$$

The discrete operator L_h can be written as an $n \times n$-tridiagonal matrix

$$L_h = \begin{bmatrix} 2S(t) & R(t) & & & \\ R(t) & 2S(t) & R(t) & & \\ & & \ddots & & \\ & & R(t) & 2S(t) & R(t) \\ & & & R(t) & S(t) - it \end{bmatrix} \tag{3.8}$$

with

$$R(t) = -1 - \frac{t^2}{6}, \qquad S(t) = 1 - \frac{t^2}{3}, \qquad t = hk$$

and

$$r_j = h(f, \phi_j), \tag{3.9}$$

where $\phi_j \in S_h(\Omega)$ is the usual hat-function.

REMARK 3. The product $t = kh$ is a measure of the number of elements per wavelength (of the exact solution). In particular, if the stepwidth is such that $t = \frac{\pi}{l}$ for integer l then exactly l elements are placed on one half-wave of the exact solution.

Discrete wavenumber and Green's function

The fundamental system of equation (3.7) is

$$F_h = \left\{ e^{-ik'x}, e^{ik'x} \mid x \in \left\{ \frac{j}{n}; j = 0, 1, \ldots, n \right\} \right\}, \tag{3.10}$$

where k' is a parameter yet to be determined. To this end, we solve any of the "interior" equations in the point $x_j = j/n,\ 1 < j < n$:

$$R(t)e^{ik'(j-1)h} + 2S(t)e^{ik'jh} + R(t)e^{ik'(j+1)h} = 0. \tag{3.11}$$

With

$$\lambda = e^{ik'h},$$

equation (3.11) has the solutions

$$\lambda_{1,2} = -\frac{S(t)}{R(t)} \pm \sqrt{\frac{S^2(t)}{R^2(t)} - 1} = \begin{cases} (*) \text{ complex conjugate}, & \text{if } \left|\frac{S(t)}{R(t)}\right| < 1, \\ (**) \text{ real}, & \text{if } \left|\frac{S(t)}{R(t)}\right| \geq 1. \end{cases} \tag{3.12}$$

From the definition of λ, we see that the *discrete wave number* k' is either real (in case (*)) or pure complex (case (**)). Physically, case (*) describes a propagating wave whereas case (**) describes a decaying wave [3]. For $h \leq \sqrt{12}/k$, one obtains always the complex conjugate solution, case (*).

The discrete wavenumber k' can be formally computed in terms of $t = kh$. From equation (3.12), case $(*)$, we get

$$\cos(k'h) = -\frac{S(t)}{R(t)}, \tag{3.13}$$

and hence,

$$k' = \frac{1}{h} \arccos\left(-\frac{S(t)}{R(t)}\right). \tag{3.14}$$

Consider the Taylor expansion

$$\begin{aligned} k'h &= \arccos\left(-\frac{S(t)}{R(t)}\right) \\ &= kh - \frac{(kh)^3}{24} + \frac{3(kh)^5}{640} + O\left((kh)^7\right). \end{aligned}$$

Hence, for fixed k,

$$k' = k - \frac{k^3h^2}{24} + 0(k^5h^4). \tag{3.15}$$

From the fundamental system F_h, the discrete Green's function is constructed (see [18,19] for details). We obtain

$$G_h(x,s) = \frac{1}{h\sin k'h} \begin{cases} \sin k'x\,(A\sin k's + \cos k's), & x \le s, \\ \sin k's(A\sin k'x + \cos k'x), & s \le x \le 1, \end{cases} \tag{3.16}$$

with

$$A = \frac{t^2 \sin k' \cos k' + i\sqrt{12}\sqrt{12-t^2}}{12 - t^2\cos^2 k'}. \tag{3.17}$$

Obviously, $|A|$ is bounded independently of k if $t = hk \le \alpha < \sqrt{12}$.

The discrete solution $u_h(x_h) = h\sum_{j=1}^{n} G_h(x_h, s_j) r_h(s_j)$ is

$$\begin{aligned} u_h(x_l) = \frac{1}{h\sin k'h} \Bigg(&\cos k'hl \sum_{j=1}^{l} r_j \sin k'hj + \\ &\sin k'hl \sum_{j=l+1}^{n} r_j \cos k'hj + A \sin k'hl \sum_{j=1}^{n} r_j \cos k'hj \Bigg), \end{aligned} \tag{3.18}$$

for $0 \le l \le n$.

3.3. Inf-sup-Stability Condition and Preasymptotic Error Estimate

In this section, we compute the Babuška-Brezzi stability constant of finite element solutions on uniform mesh using the discrete Green's function. Existence-uniqueness of the FE-solution then follows with an assumption on the magnitude of hk only. We then show discrete stability with respect to L^2-data and proceed to an H^1-estimate of the finite element error.

Discrete Babuška-Brezzi constant and stability

The discrete inf-sup constant is of the same order in k as the constant on the full space.

THEOREM 4. *Let $V_h = S_h[0,1) \subset H^1(0,1)$, and let $\mathcal{B} : V_h \times V_h \to \mathbb{C}$ be the sesquilinear form defined by equation (2.8). Then, if $hk \le 1$, the Babuška-Brezzi stability condition*

$$\inf_{u\in V_h} \sup_{v\in V_h} \frac{|\mathcal{B}(u,v)|}{|u|_1 |v|_1} = \gamma_h > 0 \tag{3.19}$$

holds and there exist positive constants C_1 and C_2, not depending on k or h such that

$$\frac{C_1}{k} \leq \gamma_h \leq \frac{C_2}{k}.$$

PROOF. The proof is similar to the infinite-dimensional case (see Appendix B). ∎

REMARK 4. We recapitulate that, for $f \in L^2(0,1)$, both approximability (Theorem 2) and the discrete stability condition hold under the assumption the hk is sufficiently small. It then follows from a fundamental theorem [3, p. 187] that the FE-solution exists and is uniquely determined. We emphasize that this result is obtained by restricting the magnitude of hk only (compare to the restriction of hk^2 to show existence-uniqueness in Theorem 3!).

A stability condition for the finite element solution with respect to L^2-data is given in the following lemma.

LEMMA 3. *Let $u_{fe} \in V_h$ be the finite element solution to the variational problem (2.8) for given data $f \in L^2(0,1)$. Then, if h is small such that $hk \leq 1$, there exists a constant C not depending on h and k such that*

$$|u_{fe}|_1 \leq C\|f\|.$$

PROOF. Since u_{fe} is piecewise linear, we have

$$\|u'_{fe}\| = \left(h\sum_{i=1}^{n}\left(d^i u_{fe}\right)^2\right)^{1/2}.$$

Write $u_h := u_{fe}|_{X_h}$ in terms of the discrete Green's function as

$$u_i = h\sum_{j=1}^{n} G_{ij} r_j,$$

then

$$d^i u = h\sum_{j=1}^{n} d^i G_{.j} r_j$$

and

$$|d^i u| \leq \|d^i G\|\,\|r\| \tag{3.20}$$

holds with

$$\|d^i G\| = \left(h\sum_{j=1}^{n}\left(d^i G_{.j}\right)^2\right)^{1/2}, \quad \|r\| = \left(h\sum_{j=1}^{n} r_j^2\right)^{1/2}. \tag{3.21}$$

From $r_j = h(f,\phi_j)$ it is easy to see that there exists a constant C_1 such that

$$\|r\| \leq C_1 h^2 \|f\|.$$

The derivatives of the Green's function are

$$d^i G_{.l} = \frac{1}{h^2 \cos\frac{k'h}{2}} \begin{cases} \cos\left(\frac{k'h}{2}(2i-1)\right)\left(A\sin k's_l + \cos k's_l\right), & i \leq l, \\ \sin k's_l \left(A\cos\left(\frac{k'h}{2}(2i-1)\right) - \sin\left(\frac{k'h}{2}(2i-1)\right)\right), & i \geq l. \end{cases}$$

Obviously, $h^2\left|d^i G_{.l}\right|$ is bounded provided that $k'h \leq \alpha < \pi$. From the Taylor series expansion of $k'h$, equation (3.15), we conclude that such α exists for sufficiently (say, $kh < 1$) small kh. Hence there is a constant C_2 such that

$$\forall i,j: \quad \left|d^i G_{.j}\right| \leq \frac{C_2}{h^2}.$$

Then also

$$\forall i: \quad \|d^i G\| \le \frac{C_2}{h^2}$$

and the statement follows from equation (3.20) with $C = C_1 C_2$. The proof is completed. ∎

We are now in a position to state the error estimate.

THEOREM 5. *Let $u \in H^2(0,1)$ be the exact solution of the variational problem (2.8) with data $f \in L^2(0,1)$ and let $u_{fe} \in S_h[0,1)$ be the finite element solution of (2.8). Then, if $hk \le 1$, the estimate*

$$|u - u_{fe}|_1 \le \left(\frac{hk}{\pi} + C\left(\frac{hk}{\pi}\right)^2 (1+k)\right) \|f\| \tag{3.22}$$

holds with a constant C not depending on h and k.

PROOF. Let $u_I \in V_h = S_h[0,1)$ be the interpolant of u and define $z \in V_h$ by $z := u_{fe} - u_I$. Then, by V_h-orthogonality of the error and linearity of the form $\mathcal{B}$,

$$\forall v \in V_h: \quad \mathcal{B}(u - u_I, v) = \mathcal{B}(z, v).$$

On the other hand, it is easy to see by partial integration that $((u-u_I)', v') = 0$ for $v \in V_h$, and therefore

$$\mathcal{B}(u - u_I, v) = k^2 (u - u_I, v).$$

Hence, z is a solution of $\mathcal{B}(z,v) = k^2\,(u - u_I, v)$ for all $v \in V_h$, and from Lemma 3 we have the estimate

$$|z|_1 \le Ck^2 \|u - u_I\|.$$

Then, by triangular inequality,

$$\begin{aligned} |u - u_{fe}|_1 &\le |u - u_I|_1 + |z|_1 \\ &\le |u - u_I|_1 + C\,k^2 \|u - u_I\|. \end{aligned} \tag{3.23}$$

We now invoke the approximation properties of the space V_h from Lemma 2 to obtain

$$|e|_1 \le \left(\frac{h}{\pi} + C\frac{k^2 h^2}{\pi^2}\right) |u|_2,$$

and the proposition follows from Lemma 1. ∎

COROLLARY 2. *If $hk \le 1$, then for $k > 1$*

$$|u - u_{fe}|_1 \le Ck \inf_{v \in S_h} |u - v|, \tag{3.24}$$

where C does not depend on h, k.

PROOF. Continue inequality (3.23) as

$$|u - u_{fe}|_1 \le (1 + C_1 k^2 h)|u - u_I|_1 \le Ck \inf_{v \in V_h} |u - v|_1$$

taking, e.g., $C = (1 + C\alpha)$ with $hk \le \alpha \le 1$. ∎

3.4. Comments

If the exact solution to the Helmholtz equation is a sinusoidal wave with frequency k, i.e., $u = A \sin kx + B \cos kx$ where A, B do not depend on k, then there are constants C_1, C_2 such that

$$\frac{C_1}{k} \le \frac{|u|_2}{|u|_1} \le \frac{C_2}{k}.$$

In this case, the estimate of Theorem 5 leads to

$$\tilde{e}_1 := \frac{|u - u_{fe}|_1}{|u|_1} \le C_1 hk + C_2 k^3 h^2, \tag{3.25}$$

i.e., the relative error in H^1-norm is bounded by k^3h^2. The first term in equation (3.25) is the approximation error. This is a local property that can be found from the analysis on any element separately.

The second term is due to numerical pollution [13]. It is a global property of the finite element solution to Helmholtz problems. Note that the pollution term is of the same size as the phase lead of the finite element solution (see [8] for a detailed discussion of this aspect). The topic of numerical pollution in the context of *a posteriori* error estimation is addressed in [20–22].

Note that the preasymptotic estimate in the theorem is a generalization of the asymptotic statement in Theorem 3. Indeed, taking out kh in equation (3.22), we directly get equation (3.6) from Theorem 3

$$|u - u_{fe}|_1 \le C|u - u_I|_1 \le C\, h\, (1 + k)\, \|f\|$$

if k^2h is small. Both error estimates hence lead to the conclusion that the stability constant C_s does not depend on k if k^2h is bounded. We will show by numerical experiment that this conclusion is sharp, i.e., the constant C_s grows with k if k^2h is not restricted.

The assumption of uniform mesh is due to technical necessities of the proofs for Theorems 4 and 5. All statements of this section should hold for quasiuniform mesh as well.

4. NUMERICAL EVALUATION

Throughout this section, we consider the variational problem (2.8) with constant right hand side $f(x) \equiv -1$.

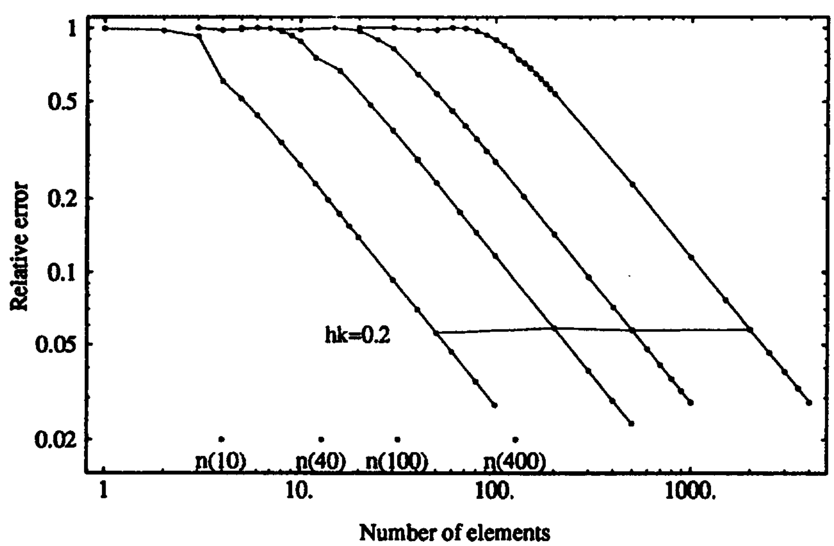

Figure 1. Relative error of the best approximation in H^1-seminorm and predicted critical numbers of DOF for $k = 10$, $k = 100$ and $k = 400$.

Error of the best approximation

Consider in Figure 1 log-log-plots of the relative error $\tilde{e}_a := |u - u_I|_1/|u|_1$ of the best approximation in H^1-seminorm for different k. All error curves decrease with constant slope of -1. Note that the error stays at 100% on coarse mesh and starts to decrease at a certain meshsize. We are interested in the point where the descent starts. More precisely, we seek the critical number of degrees of freedom according to the following definition.

Define—for any fixed k and f—the *critical number of degrees of freedom (DOF)* as the minimal number $N(k,f)$ of DOF for which

1. $\tilde{e}(n,k) < 1$ and
2. $\tilde{e}(n,k)$ is monotone decreasing with respect to n

for $n > N(k,f)$.

For the best approximation, the critical number of DOF is determined by the rule that the stepwidth of interpolation by piecewise linears should be smaller than one half of the wavelength of the exact solution, i.e., $hk < \pi$. The critical point n_o, computed accordingly from

$$n_o = \left[\frac{k}{\pi}\right] \tag{4.1}$$

is plotted for different k. It coincides well with the start of convergence on all curves.

The figure also shows that the error of the best approximation is controlled by the magnitude hk. For illustration, the points that are computed from $hk \equiv 0.2$ are connected. The connecting line does neither increase nor decrease significantly with the change of k.

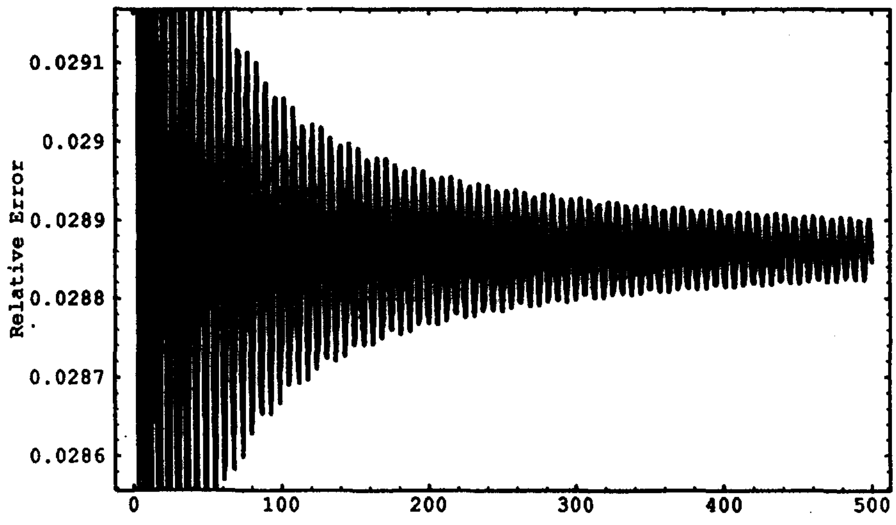

Figure 2. Relative error of the best approximation in H^1-seminorm computed for $k = 1 \ldots 500$ with stepwidth h determined by $hk = 0.1$.

For more detailed observation, the relative error of the best approximation, computed for all integer k from 1 to 500 and for $hk \equiv 0.1$, is plotted in Figure 2. The error oscillates with decaying amplitude around the horizontal line

$$|\tilde{e}_a|_1 = 0.02887.$$

With $|u|_2/|u|_1 \approx k$ for sufficiently large k, the upper estimate from Lemma 2 is

$$|\tilde{e}_a|_1 \leq \frac{0.1}{\pi} = 0.03183.$$

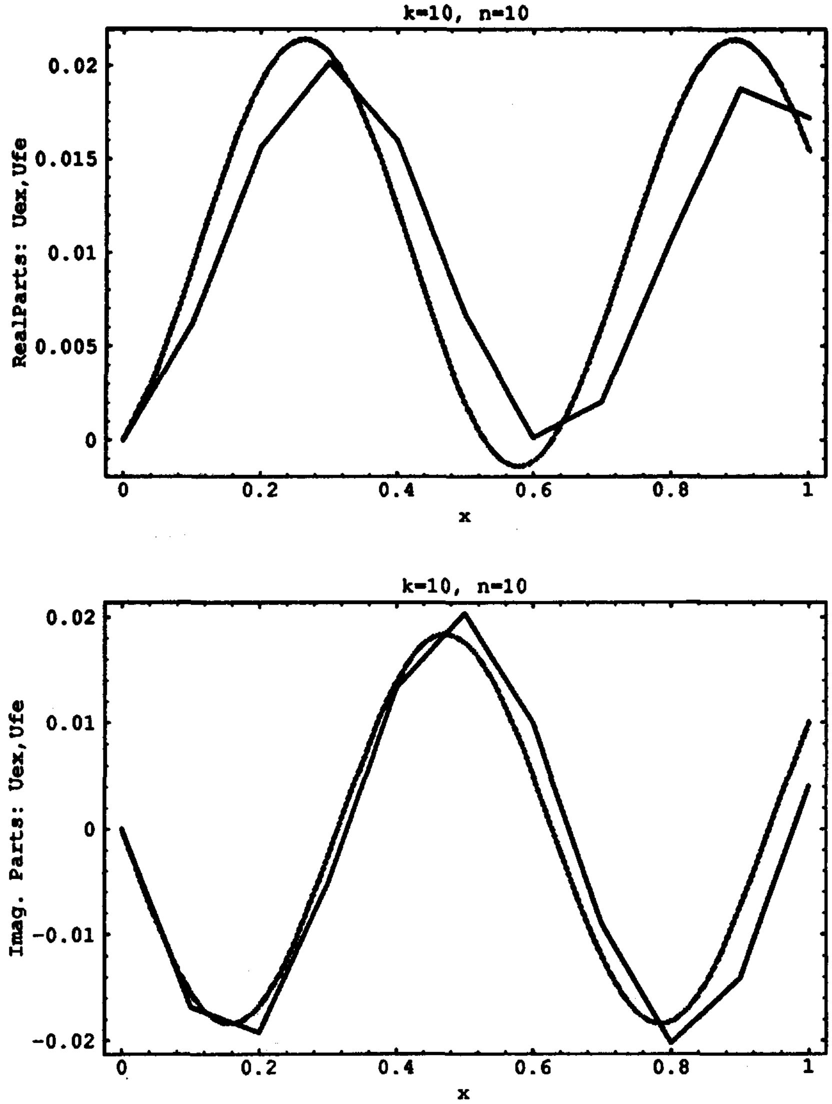

Figure 3. Phase lead of the Finite element solution for $k = 10$, $n = 10$.

Discrete wavenumber

Unlike the best approximation, the finite element solution is, in general, not in phase with the exact solution. The discrete solution has a phase lead with respect to the exact solution. This is shown in Figure 3, where the real and imaginary parts of both solutions are plotted for $k = 10,\ hk = 1$.

On uniform mesh, the relation

$$\cos k'h = -\frac{S(t)}{R(t)},$$

where $t = hk$ and the right-hand side is a rational function of t—equation (3.13), is used for computation of the discrete wavenumber that governs the periodicity of the finite element solution. In Figure 4, the functions $y_1 = -S(t)/R(t)$, $y_2 = \cos t$ and $|y_3| = 1$ are plotted. At $t_o = \sqrt{12}$, the function y_1 reaches absolute value 1; the numerical solution switches from the propagating case to the decaying case. The value t_o corresponds to a *cutoff frequency* for the numerical solution [23].

For fixed k, the convergence $k' \to k$ is visualized by $\cos k'h \to \cos t = \cos kh$ as $h \to 0$. The curves begin to deviate significantly at about $hk = 1$.

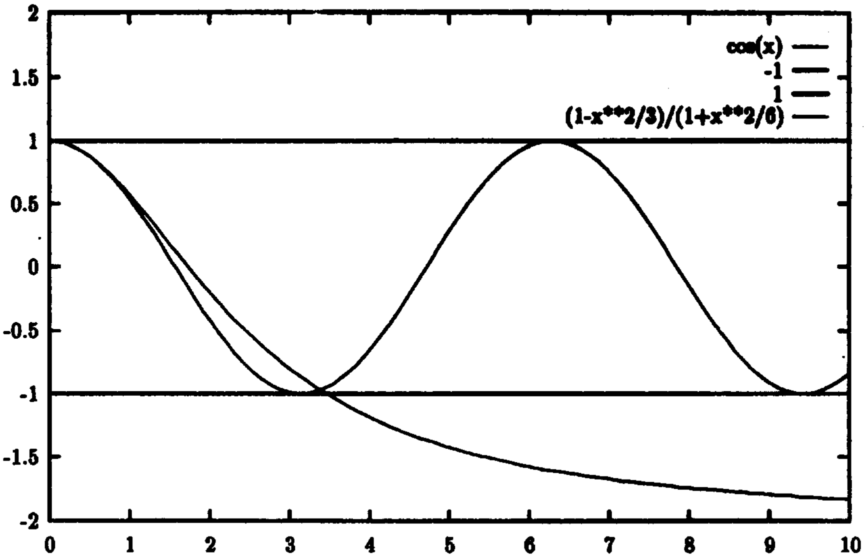

Figure 4. Convergence of discrete to exact wavenumber via comparison of $\cos k'h = -S(x)/R(x)$ to $\cos(x)$ for $x = kh$.

Error of the finite element solution

In Figure 5, the relative error of the finite element solution in H^1-seminorm is plotted for different k.

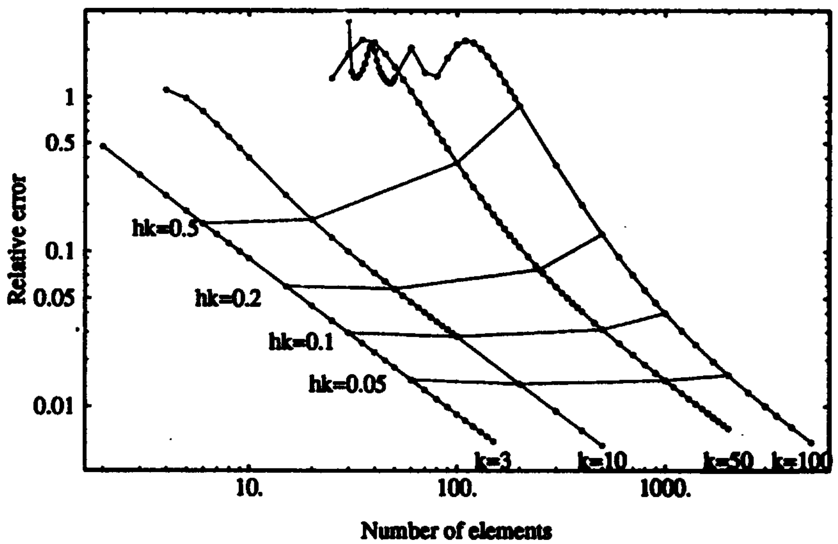

Figure 5. Relative error in $H1$-seminorm: Finite element solutions for $k = 3$, $k = 10$, $k = 50$ and $k = 100$.

For low k ($k = 3$, $k = 10$), the finite element solution converges as the best approximation. For high k, the relative error oscillates above 100% before a critical value of degrees of freedom is reached. The decrease then first occurs with a rate greater than -1 in the log-log-scale but becomes -1 for small h. The relative error generally grows with k along lines $hk \equiv$ const. Unlike the error of the best approximation, the error of the finite element solution is not controlled by the magnitude of hk—see also Figure 6 and Table 1.

Asymptotic stability and quasioptimality

In Figure 7, the relative error of the finite element solution and the relative error of the best approximation are displayed in one plot. To enhance the quasioptimal stability estimate of

Table 1. Number of elements per wavelength needed for accuracy of 10% in H^1-seminorm.

k	100	200	300	400	600	800	1000
# of elements	38	57	63	82	94	107	120

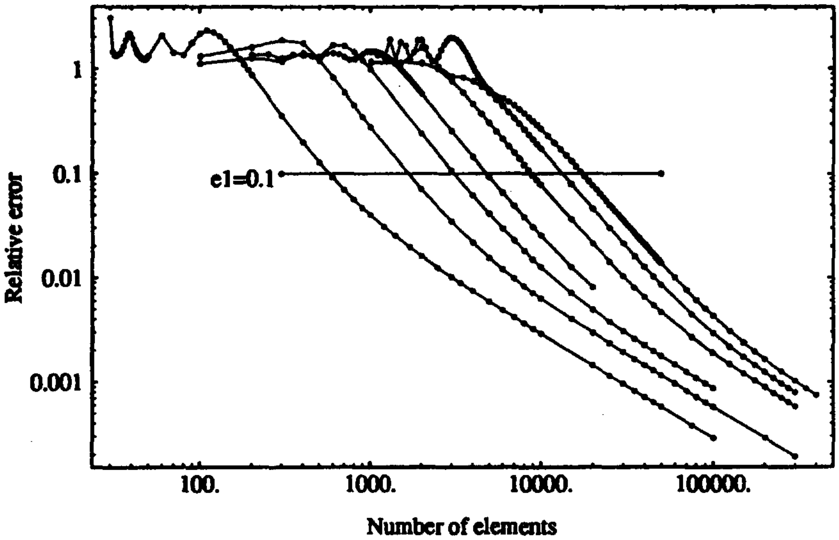

Figure 6. Relative error in $H1$-seminorm: Finite element solutions for $k = 100$, $200, 300, 400, 600, 800$ and $k = 1000$.

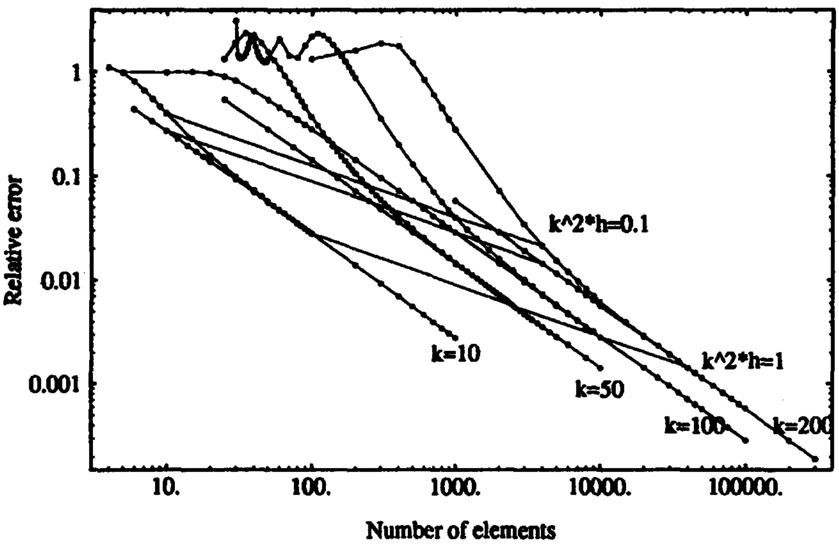

Figure 7. Relative error of the finite element solution and the approximation in H^1-seminorm for $k = 10, 50, 100$ and $k = 200$. Meshes with $k^2h = 0.1$ or $k^2h = 1$, respectively, are connected on both curves.

Theorem 3, the lines $k^2h = \alpha \equiv$ const are plotted for $\alpha = 1$ and $\alpha = 0.1$. We observe that along these lines the ratio of the errors does not depend on k (the distances between both curves in the log-log-plot do not grow). This is exactly in accord with the quasioptimal estimate stating that the ratio e_{fe}/e_{ba} is bounded by a stability constant C_s independently on k, h. In Figure 8, the ratio e_{fe}/e_{ba}, computed with the restriction $k^2h = 1$, is plotted for k from 1 to 200. Obviously, the ratio does neither decrease nor grow with increasing k. On the other hand, the error ratio *does depend* on k on all lines $hk^{\beta} = \alpha$ with $\beta < 2$. In particular, C_s is increasing with k on the line defined by $hk = 1$, as shown in Figure 9.

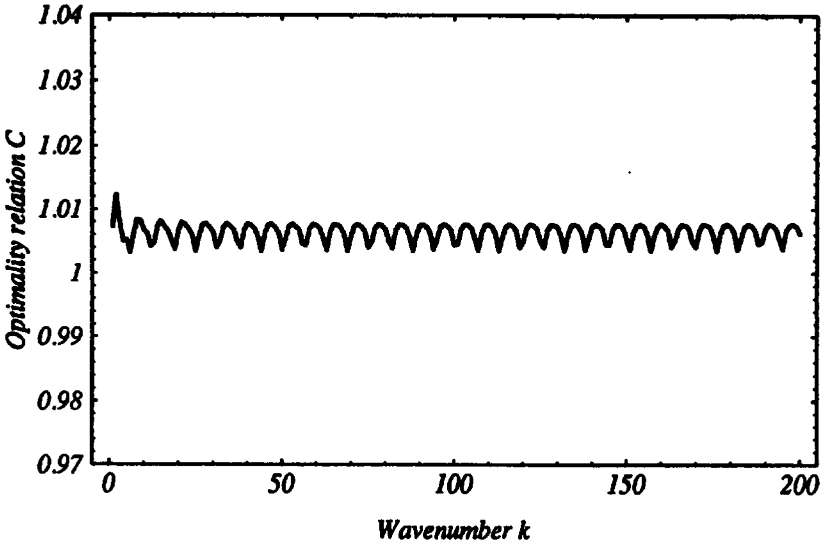

Figure 8. Relation e_{fe}/e_{ba} of the finite element error to the minimal error H^1-seminorm with $k^2h = 1$.

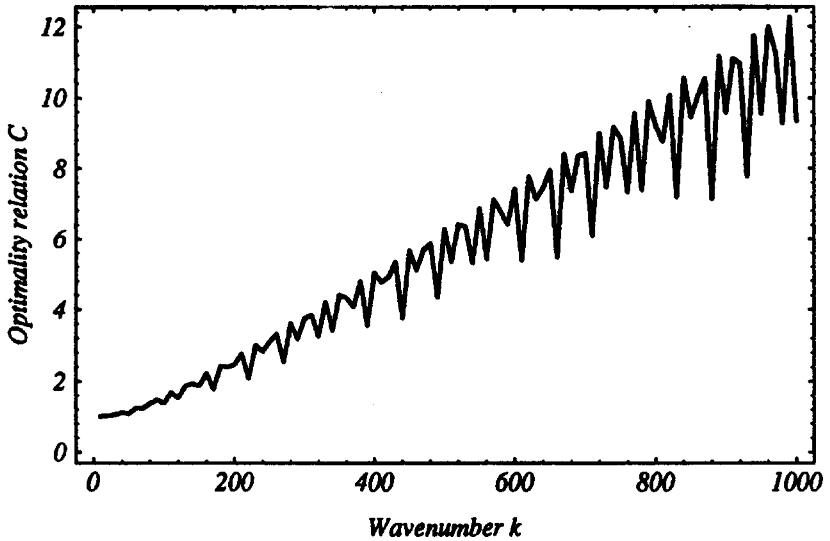

Figure 9. Ratio e_{fe}/e_{ba} of the finite element error to the minimal error H^1-seminorm with $hk = 0.1$.

Preasymptotic stability and error estimate

We have seen that the assumption on k^2h is necessary for quasioptimal convergence in H^1-seminorm of the finite element solution. However, it is not necessary to bound this ratio for the practical purpose of limiting the error of the FE-solution at finite range. Indeed, C_s grows with k on the line of constant relative error of the FE-solution (Figure 10).

According to Theorem 5, the relative error is bounded at any range by the magnitudes of h^2k^3 and hk. In Figure 11, the relative error of the finite element solution for k from 1 to 1000 on meshes with $h = 1/(k^{3/2})$ is shown. We observe:

- For low k ($1 \leq k \leq 50$), the relative error decreases rapidly with k. In this range, the FE-solution is still close to the best approximation ($hk^2 = 5.48$ for $k = 30$) and hence, the term hk is the significant member in the estimate (3.40).
- For large k ($k \geq 100$), the error is bounded by $\tilde{e} = 0.05$. The term h^2k^3 is leading in estimate (3.40).

Consider the effect on the results of applied computations. To this end, we write the estimate of Theorem 5 in the form

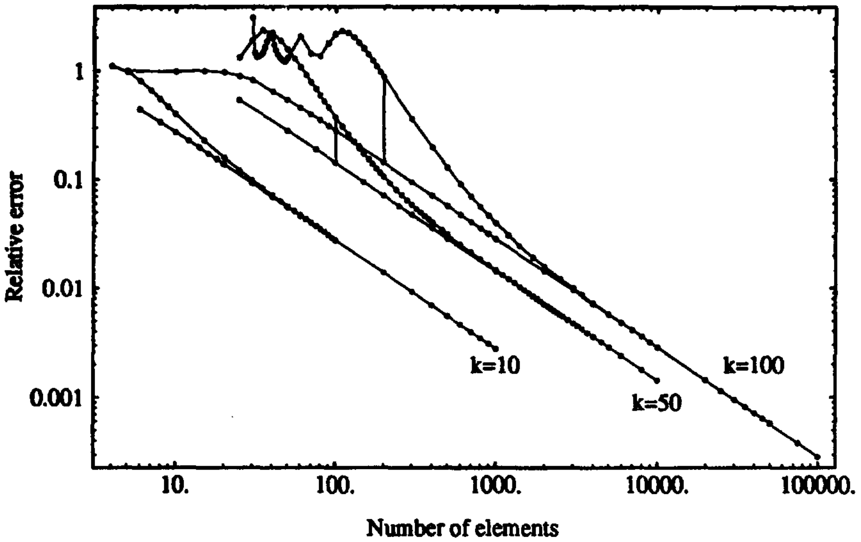

Figure 10. Stability constant C_s (vertical lines) at relative error of $\approx$ 20% for $k = 10$, $k = 50$ and $k = 100$.

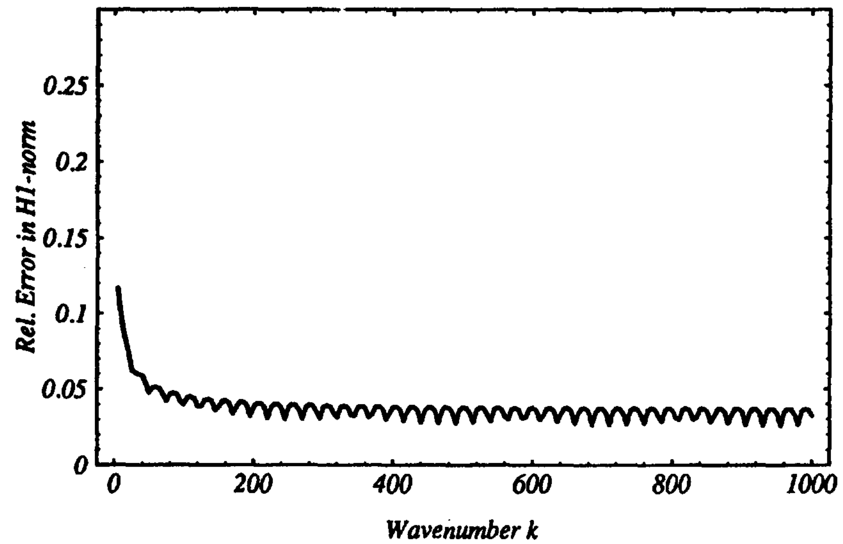

Figure 11. Relative error of the finite element solution in H^1-seminorm with constraint $h^2k^3 = 1$ for $k = 1, 1000, 1$.

$$|e|_1 \leq \left(\alpha + C(1+k)\alpha^2\right) \|f\| \tag{4.2}$$

with $\alpha := hk/\pi$. Let $\alpha = 0.1$, i.e., the wavelength is resolved by 20 elements. Then for $k = 10$, we have $k\alpha^2 = 0.1$: both terms in equation (4.2) are of the same magnitude, and hence, the phase lead does not affect the error significantly. Consequently, no negative effects should be observed in benchmark tests. However, for high wavenumber (say, $k = 100$) the second member equals 1 for the same resolution $\alpha = 0.1$ and hence dominates the estimate. The pollution effect is still more significant for lower resolutions like $\alpha = 0.2$ or $\alpha = 0.5$ (cited as "acceptable resolution" or "limit of resolution," respectively, in [3]). For $k = 10$, the magnitudes $\alpha = 0.2$ and $k\alpha^2 = 0.4$ are still of the same order for acceptable resolution but differ considerably for the limit of resolution ($\alpha = 0.5$ and $k\alpha^2 = 2.5$). For high wavenumber ($k = 100$), the second member of the estimate is clearly dominating for both resolutions: we have $\alpha = 0.2$ vs. $k\alpha^2 = 4$ and, for the limit of resolution, $\alpha = 0.5$ vs. $k\alpha^2 = 25$.

Finally, we demonstrate that also the critical number of DOF for the finite element error is governed by the magnitude of h^2k^3. In Figure 12, the numbers

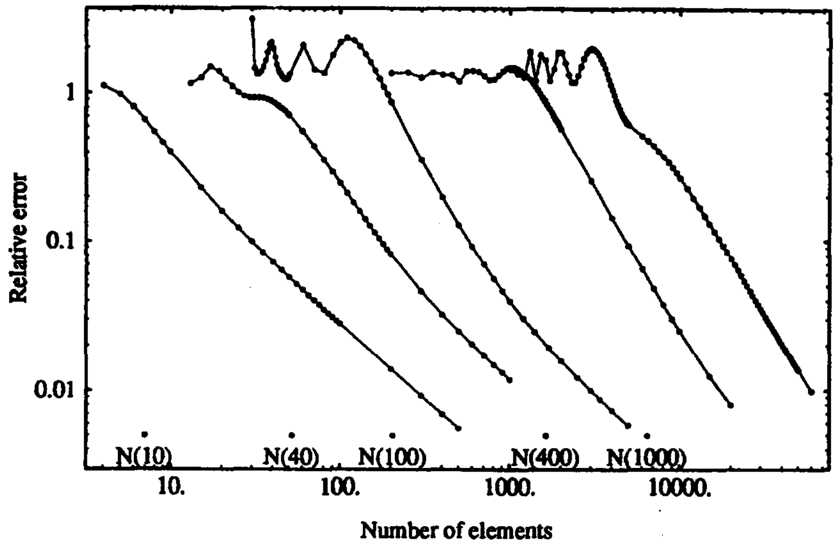

Figure 12. Relative error of the finite element solution in H^1-seminorm and predicted position of the "knee" (critical Numbers of DOF) for $k = 10, 40, 100, 400$ and $k = 1000$.

$$N_o = \sqrt{\frac{k^3}{24}} \tag{4.3}$$

are plotted for different k. The predicted critical number of DOF is close to the start of convergence of the finite lement solution. The formula (4.3) is motivated as follows. Assume that the solutions are given by $u = \sin kx$ and $u_h = u_{fe}|_{X_h} = \sin k'x_h$ and consider the error in the L_∞-norm. Then, if the phase lead $k' - k$ is smaller than $\pi/2$, the maximal difference of amplitudes $|\sin kx_h - \sin k'x_h|$ occurs at the end of the interval $[0,1]$. For $x = 1$, the error of the finite element solution is

$$|\sin k - \sin k'| = 2\left|\cos\frac{k+k'}{2}\right|\left|\sin\frac{k-k'}{2}\right|.$$

Since $\|\sin kx\|_\infty = 1$ for sufficiently large k, the relative error in L^∞-norm is smaller than 1 if

$$\left|\sin\frac{k-k'}{2}\right| \le \frac{1}{2}$$

or, equivalently,

$$k - k' \le \frac{\pi}{3} \approx 1.$$

With this, equation (4.3) follows from the Taylor expansion equation (3.21).

5. CONCLUSIONS

The numerical solution of the Helmholtz equation with the h-version of the FEM is studied on a one-dimensional model problem. New analytical statements that hold in the preasymptotic range of discretisation are shown. The analytical study is completed with results of computational experiments.

This investigation of the Galerkin finite element method on a one-dimensional model problem for the Helmholtz equation reveals:

- The finite element solution is stable given only restrictions on the magnitude of hk.
- The relative error in H^1-seminorm of best approximation in the finite element subspace is controlled by a term of order hk. If k^2h is small, then the finite element solution is

quasioptimal, i.e., equivalent to the best approximation; the equivalence relation does not depend on k.

- In the preasymptotic range, the relative error in H^1-norm of the finite element solution is governed by the term h^2k^3, and hence, can be controlled restricting this magnitude.
- The Babuška-Brezzi stability constant is of order k^{-1} both in the continuous and the discrete case.
- The restriction of hk^2 is not only sufficient, but also necessary for quasioptimality of the finite element solution in H^1-norm.

If hk^2 is small, then the finite element solution is in the *asymptotic* range of convergence where it is close to the interpolant of the exact solution and hence is quasioptimal, i.e., the finite element error is proportinal (independently of k) to the interpolation error. In the *preasymptotic* range, the difference between the finite element solution and the interpolant (the *phase lead* of the finite element solution) is the dominant part of the finite element error.

In Part II, results are presented for the *h-p*-version of the Galerkin FEM for Helmholtz problems.

APPENDIX A

PROOF OF THEOREM 3. Denote $e := u - u_{fe}$. Then e lies in the Hilbert space $V \subset H^1(0,1)$ and, consequently (cf. Remark 3), there exists $z \in V$ such that

$$\forall v \in V: \quad \mathcal{B}(v,z) = (v,e).$$

In particular, $\mathcal{B}(e,z) = (e,e)$ for $v = e$.

Further, the error is $\mathcal{B}$-orthogonal to the discrete test space $V_h := S_h[0,1]$:

$$\forall w \in V_h: \quad \mathcal{B}(e,w) = 0.$$

Then, for all $w \in V_h$,

$$\begin{aligned}\|e\|^2 = (e,e) &= \mathcal{B}(e, z-w)\\ &= \int e'\,(\overline{z-w})' - k^2 \int e\,(\overline{z-w}) - ike(1)\,(\overline{z(1)-w(1)})\\ &\le \|\,(z-w)'\|\,\|e'\| + k^2\|z-w\|\,\|e\| + k|z(1)-w(1)|\,|e(1)|.\end{aligned}$$

Apply the inequality $|v(1)| \le \sqrt{2}\|v\|^{1/2}\|v'\|^{1/2}$ which is true for all $v \in V$ to obtain

$$\begin{aligned}k|z(1)-w(1)|\,|e(1)| &\le 2k\|\,(z-w)'\|^{1/2}\|e'\|^{1/2}\|z-w\|^{1/2}\|e\|^{1/2}\\ &\le k^2\|z-w\|\,\|e\| + \|\,(z-w)'\|\,\|e'\|,\end{aligned} \tag{A.1}$$

where the inequality $2ab \le a^2 + b^2$ has been applied. This gives, for all $w \in V_h$,

$$\|e\|^2 \le 2\left(\|\,(z-w)'\|\,\|e'\| + k^2\|z-w\|\,\|e\|\right).$$

In particular, we may apply Lemmas 1 and 2 for $w = z_I \in V_h$ (the piecewise linear interpolant of z) to obtain

$$\begin{aligned}\|e\|^2 &\le \left(\|\,(z-z_I)'\|\,\|e'\| + k^2\|z-z_I\|\,\|e\|\right)\\ &\le 2\left((1+k)\frac{h}{\pi}\|e'\|\,\|e\| + k^2\frac{h^2}{\pi^2}(1+k)\,\|e\|^2\right).\end{aligned}$$

Divide both sides of the inequality above by the common factor $\|e\|$, then

$$\|e\| \le C_1\,(1+k)\,h\,\|e'\| \tag{A.2}$$

holds with

$$C_1 := \frac{2}{\left(1 - 2(1+k)\frac{k^2h^2}{\pi^2}\right)\pi}$$

under the assumption that k, h are such that the denominator of C_1 is positive.

Next, from $\mathcal{B}$-orthogonality of the error to elements from V_h, we have

$$\mathcal{B}(e,e) = \mathcal{B}(e, u - u_{fe}) = \mathcal{B}(e,u),$$

and hence,

$$\forall v \in V_h: \quad \mathcal{B}(e,e) = \mathcal{B}(e, u - v).$$

Thus, for all $v \in V_h$

$$\int e'\overline{e}' - k^2 \int e\bar{e} - ik|e(1)|^2 = \int e'(\overline{u-v})' - k^2 \int e(\overline{u-v}) - ike(1)(\bar{u}(1) - \bar{v}(1))$$

and therefore,

$$\begin{aligned}
\|e'\|^2 \leq\; & k^2\|e\|^2 + k|e(1)|^2 + \|e'\|\,\|(u-v)'\| + k^2\|e\|\,\|u-v\| + k|e(1)|\,|u(1) - v(1)| \\
\leq\; & k^2\|e\|^2 + 2k\|e'\|\,\|e\| + 2\|e'\|\,\|(u-v)'\| + 2k^2\|e\|\,\|u-v\|,
\end{aligned}$$

where the terms in $x = 1$ have been estimated as in (A.1). We now use the ε-inequality to get the estimates

$$\begin{aligned}
2k\,\|e'\|\,\|e\| &\leq \frac{1}{4}\,\|e'\|^2 + 4k^2\|e\|^2, \\
2\,\|e'\|\,\|(u-v)'\| &\leq \frac{1}{4}\,\|e'\|^2 + 4\|(u-v)'\|^2, \\
2k^2\,\|e\|\,\|u-v\| &\leq k^2\,\|e\|^2 + k^2\|u-v\|^2.
\end{aligned}$$

Introducing these estimates into the inequality leads to

$$\forall v \in V_h: \quad \|e'\|^2 \leq 6k^2\|e\|^2 + \frac{1}{2}\|e'\|^2 + 4\|(u-v)'\|^2 + k^2\|u-v\|^2. \tag{A.3}$$

Then, using the intermediary result (A.2) and the approximation results from Lemma 2 for $v = u_I$, we get

$$\frac{1}{2}\|e'\|^2 \leq 6k^2(1+k)^2C_1^2h^2\|e'\|^2 + 4\left(\frac{h}{\pi}\right)^2(1+k)^2\|f\|^2 + k^2(1+k)^2\left(\frac{h}{\pi}\right)^4\|f\|^2,$$

and hence,

$$\left(\frac{1}{2} - 6k^2(1+k)^2C_1^2h^2)^{1/2}\right)\|e'\| \leq \left(\frac{2}{\pi}\right)\left(1 + \left(\frac{hk}{2\pi}\right)^2\right)^{1/2} h(1+k)\|f\|$$

and the statement of the theorem follows. The proof is completed.

To prove the corollary, introduce equation (3.4) from Lemma 2 to (A.3).

APPENDIX B

Proof of Theorem 4. We show that for any given $u \in V_h$ there exists some $v \in V_h$ such that

$$|\mathcal{B}(u,v)| \geq \frac{C}{k}\|u'\|\,\|v'\|.$$

Hence, let $u \in V_h$ be given and define $v := u + z$ where $z \in V_h$ is a solution of the variational problem

$$\forall w \in V_h : \quad \mathcal{B}(w, z) = k^2(w, u). \tag{B.1}$$

Since V_h is a Hilbert space, the solution of (B.1) exists and is uniquely defined. As in the continuous case, we will now prove that

$$|u|_1 \geq \frac{C}{k}|v|_1$$

using the Green's function representation of z:

$$z_i = z_h(x_i) = h\sum_{j=1}^{n} G_{ij} r_j, \tag{B.2}$$

where

$$G_{ij} := G_h(x_i, s_j); \quad r_j := r_h(s_j).$$

Summation by parts in equation (B.2) yields

$$z_i = H_{in} r_n - H_{i1} r_o - h\sum_{j=1}^{n} H_{ij} d^j r \tag{B.3}$$

with

$$D^j H_{i\cdot} = G_{ij}, \qquad j = 1, \dots n-1. \tag{B.4}$$

Since the mesh function H is defined by equation (B.4) up to a constant, we are free to choose

$$H_{i1} = 0.$$

Let us now take the left differences of z_h in some fixed point $i = l$:

$$d^l z = d^l H_{\cdot n} r_n - h\sum_{j=1}^{n} d^l H_{\cdot j} d^j r. \tag{B.5}$$

Then, applying the Schwarz inequality, we obtain the estimate

$$\begin{aligned} |d^l z| &\leq |d^l H_{\cdot n}|\,|r_n| + \|H_x\|\,|r|_1 \\ &\leq \left(|d^l H_{\cdot n}| + \|H_x\|\right)|r|_1. \end{aligned} \tag{B.6}$$

The right-hand side of the variational problem is by direct computation

$$r_j = \frac{1}{6}k^2h^2\left(u_{j-1} + 4u_j + u_{j+1}\right), \qquad j = 1, \dots n-1$$

hence,

$$|r|_1 \leq Ch^2k^2|u|_1 \tag{B.7}$$

where C is a constant of order 1. We now turn to estimation of the magnitude $|d^l H_{\cdot n}| + \|H_x\|$. From equation (B.4), we obtain after summation over j

$$H_{ij} - H_{i1} = h\sum_{l=1}^{j-1} D^l H_{i\cdot} = h\sum_{l=1}^{j-1} G_{il}$$

and consequently, since $H_{i1} = 0$,

$$H_{ij} = h\sum_{l=1}^{j-1} G_{il}. \tag{B.8}$$

Taking left differences, we obtain

$$d^i H_{\cdot j} = h \sum_{l=1}^{j-1} d^i G_{\cdot l}. \tag{B.9}$$

The derivatives (as left differences) of the discrete Greens function are

$$d^i G_{\cdot l} = \frac{1}{h \sin k'h} \begin{cases} d^i \sin k'x_h \left(A \sin k's_l + \cos k's_l\right), & x_h \le s_l, \\ \sin k's_l \left(A d^i \sin k'x_h + d^i \cos k'x_h\right), & x_h \ge s_l. \end{cases} \tag{B.10}$$

We substitute

$$d^i \sin k'x_h = \frac{2}{h} \cos\left(\frac{k'h}{2}(2i-1)\right) \sin\frac{k'h}{2},$$
$$d^i \cos k'x_h = -\frac{2}{h} \sin\left(\frac{k'h}{2}(2i-1)\right) \sin\frac{k'h}{2},$$

to obtain

$$d^i G_{\cdot l} = \frac{1}{h^2 \cos\frac{k'h}{2}} \begin{cases} \cos\left(\frac{k'h}{2}(2i-1)\right)\left(A \sin k's_l + \cos k's_l\right), & i \le l, \\ \sin k's_l \left(A \cos\left(\frac{k'h}{2}(2i-1)\right) - \sin\left(\frac{k'h}{2}(2i-1)\right)\right), & i \ge l. \end{cases} \tag{B.11}$$

Then, for $j > i+1$,

$$\begin{aligned}
\sum_{l=1}^{j-1} d^i G_{\cdot l} &= \frac{1}{h^2 \cos\frac{k'h}{2}} \left(\cos\left(\frac{k'h}{2}(2i-1)\right)\left(A \sum_{l=1}^{i} \sin k'hl + \sum_{l=1}^{i} \cos k'hl\right)\right. \\
&\quad \left. + \sum_{l=i+1}^{j-1} \sin k'hl \left(A \cos\left(\frac{k'h}{2}(2i-1)\right) - \sin\left(\frac{k'h}{2}(2i-1)\right)\right)\right) \\
&= \frac{1}{h^2 \cos\frac{k'h}{2} \sin\frac{k'h}{2}} \left(\cos\left(\frac{k'h}{2}(2i-1)\right)\right) \\
&\quad \times \left(A \sin\frac{ik'h}{2} \sin\frac{(i+1)k'h}{2} + \sin\frac{ik'h}{2} \cos\frac{(i+1)k'h}{2}\right) \\
&\quad + \left(\sin\frac{(j-1)k'h}{2} \sin\frac{jk'h}{2} - \sin\frac{ik'h}{2} \sin\frac{(i+1)k'h}{2}\right)\Bigg) \\
&\quad \times \left(A \cos\left(\frac{k'h}{2}(2i-1)\right) - \sin\left(\frac{k'h}{2}(2i-1)\right)\right)\Bigg) \\
&\le \frac{D_1}{h^2 \sin k'h},
\end{aligned}$$

since $|A|$ and hence the expression in the brackets are bounded. With the assumption that kh and hence $k'h$ is small there exists $D_2 > 0$ such that

$$\sin k'h = k'h \left(1 - \frac{k'^2 h^2}{6} \pm \cdots\right) \ge D_2 k'h,$$

then

$$\begin{aligned}
\|H_x\| &= \left(h \sum_{j=1}^{n} \left|d^i H_{\cdot j}\right|^2\right)^{1/2} \\
&= \left(h \sum_{j=1}^{n} \left|h \sum_{l=1}^{j-1} d^i G_{\cdot l}\right|^2\right)^{1/2} \\
&= h^{3/2} \left(\sum_{j=1}^{n} \left(\sum_{l=1}^{j-1} d^i G_{\cdot l}\right)^2\right)^{1/2},
\end{aligned}$$

and with the previous inequalities, we obtain

$$\|H_x\| \le h^{3/2} \left(\sum_{j=1}^{n} \frac{D_1 D_2^{-2}}{h^6 k'^2} \right)^{1/2} \le \frac{D_3 h^{3/2}}{h^{7/2} k'} \le \frac{D_3}{h^2 k'}.$$

By similar computation, we can show that for any $l, 1 \le l \le n$

$$|d^l H_{\cdot n}| = \left| h \sum_{j=1}^{n} d^i G_{\cdot j} \right| \le \frac{D_4}{h^2 k'},$$

hence,

$$\|H_x\| + \max_l |d^l H_{\cdot n}| \le \frac{D}{h^2 k'}$$

where $D = D_3 + D_4$.

Returning now to equations (B.5) and (B.7),

$$\begin{aligned} |z|_1 &= \left(h \sum_{j=1}^{n} |d^l z|^2 \right)^{1/2} \\ &\le \left(\max_{1<l<n} |d^l H_{\cdot n}| + \|H_x\| \right) |r_1|_1 \\ &\le \frac{D}{h^2 k'} C h^2 k^2 |u|_1 \\ &\le k C D \left(\frac{k}{k'} \right) |u|_1. \end{aligned}$$

From the Taylor series expansion (3.15), we see that

$$\frac{k'}{k} = 1 + \frac{k^2 h^2}{6} - \frac{3 k^4 h^4}{640} \pm \cdots$$

is bounded for sufficiently small kh. Hence, there exists a constant E not depending on h and k such that

$$|z|_1 \le E k |u|_1. \tag{B.12}$$

We then have

$$|v|_1 = |u + z|_1 \le (1 + Ek)|u|_1,$$

hence, there exists, for sufficiently large k, a constant F such that

$$|u|_1 \ge \frac{F}{k} |v|_1$$

and left inequality of the statement follows from the definition of z and the Gårdings-type inequality (2.11).

To prove the right inequality, we construct, in analogy to Section 2, a function z_o for which continuity holds with Ck^{-1}. Consider the function

$$Z(x) = \varphi(x)\, w(x),$$

where $\varphi(x) \in C^\infty(0, 1)$ and

$$w(x) = \frac{\sin k' x}{k}$$

is a fundamental solution of the discrete system equation (3.7). Let $z_o(x) \in V_h$ be the piecewise linear interpolant of $Z(x)$ on X_h. Again we assume that φ does not depend on the parameter k and is selected such that

$$\varphi(0) = \varphi(1) = \varphi'(1) = 0,$$

and there exists $\alpha > 0$ such that

$$|z_o|_1 \geq \alpha$$

independently on k. Then

$$\forall v \in V_h : \quad \frac{|\mathcal{B}(z_o, v)|}{|z_o|_1} \leq \frac{1}{\alpha}|B(z_o, v)|.$$

Turn to the estimation of $|\mathcal{B}(z_o, v)|$ (we omit the subscript o from now on):

$$\begin{aligned}\mathcal{B}(z, v) &= \int_0^1 z'v' - k^2 \int_0^1 zv \\ &= h\sum_{j=1}^{n} d^j z d^j v - \frac{k^2}{6} h \sum_{j=1}^{n} (z_{j-1} + 4z_j + z_{j+1})\, v_j\end{aligned}$$

(let formally $z_{n+1} := z_{n-1}$). Summation by parts then yields

$$\mathcal{B}(z, v) = -h \sum_{j=1}^{n} \left(D^j(d^j z) + \frac{k^2}{6}(z_{j-1} + 4z_j + z_{j+1}) \right) \bar{v}_j + \frac{1}{h}(z_{n-1}\bar{v}_n - z_o\bar{v}_o).$$

The term outside the sum is $O(h)$. Indeed, $z_o = 0$ and

$$\varphi_{n-1} = \varphi(1) - h\varphi'(1) + \frac{h^2}{2}\varphi''(1) + O(h^3).$$

Consequently, since $\varphi(1) = \varphi'(1) = 0$, we have $h^{-1}z_{n-1} = h^{-1}\varphi_{n-1}w_{n-1} = O(h)$. Hence, omitting the terms $O(h)$,

$$\mathcal{B}(z, v) = -h \sum_{j=1}^{n} \left(D^j(d^j z) + \frac{k^2}{6}(z_{j-1} + 4z_j + z_{j+1}) \right) \bar{v}_j.$$

For arbitrarily fixed j, we write the second differences as

$$\begin{aligned} D^j(d^j z) &= D^j(d^j(\varphi w)) = D^j\left((d^j\varphi)w_{j-1} + \varphi_j d^j w\right) \\ &= D^j(d^j\varphi)w_{j-1} + 2D^j\varphi d^j w + \varphi_j D^j(d^j w)\end{aligned}$$

and the weighted sum as

$$\begin{aligned} z_{j-1} + 4z_j + z_{j+1} &= (\varphi w)_{j-1} + 4(\varphi w)_j + (\varphi w)_{j-1} \\ &= w_{j-1}(\varphi_j - h\varphi_j' + O(h^2)) + 4w_j\varphi_j + w_{j+1}(\varphi_j + h\varphi_j' + O(h^2)) \\ &= \varphi_j\,(w_{j-1} + 4w_j + w_{j+1}) + 2h^2\varphi_j' w_j' + O(h^2)).\end{aligned}$$

Then, neglecting all terms that are $O(h)$ we can write

$$\begin{aligned} &D^j(d^j z) + \frac{k^2}{6}(z_{j-1} + 4z_j + z_{j+1}) = \\ &\qquad \varphi_j \left[D^j(d^j w) + \frac{k^2}{6}(w_{j-1} + 4w_j + w_{j+1}) \right] + D^j(d^j\varphi)w_{j-1} + 2D^j\varphi d^j w.\end{aligned}$$

Since w has been selected as a fundamental solution of the discrete system, the expression in square brackets vanishes. We now define the piecewise linear function u as the linear interpolant of the meshfunction u_h defined by

$$u_h(x_i) := h\sum_{j=1}^{i}\left(D^j(d^j z) + \frac{k^2}{6}(z_{j-1} + 4z_j + z_{j+1})\right).$$

Then, on the one hand,

$$|\mathcal{B}(z,v)| = \left|u(1)\bar{v}(1) - \int_0^1 u(x)\bar{v}'(x)\,dx\right| \leq (|u(1)| + \|u\|)|v|_1,$$

and on the other hand

$$\begin{aligned}\|u\| &= \left(h\sum_{i=1}^{n}\left(h\sum_{j=1}^{i-1}\left(D^j(d^j\varphi)w_{j-1} + 2D^j\varphi d^j w\right)\right)^2\right)^{1/2}\\ &= \left(h\sum_{i=1}^{n}\left(h\sum_{j=1}^{i-1}\left(-D^j(d^j\varphi)w_{j-1} + 2\left(D^{i-1}\varphi w_{i-1} - w_1 D^o\varphi\right)\right)\right)^2\right)^{1/2}.\end{aligned}$$

Making use of the smoothness of the function φ, we have for all j

$$\begin{aligned}D^j(d^j\varphi) &= \varphi''(jh) + O(h^2),\\ D^{j-1}\varphi &= \varphi'((j-1)h) + O(h),\end{aligned}$$

and we obtain

$$\|u\| \leq h\sum_{i=1}^{n}\left(hi|w|\left(\|\varphi''\|_\infty + (\|\varphi'\|_\infty + 2\|\varphi'\|_\infty + O(h))\right)^2\right)^{1/2},$$

where the function $w = k^{-1}\sin k'x$ can be estimated by

$$|w| \leq \frac{1}{k}$$

and the term $O(h)$ does not depend on k.

By similar estimates for $|u(1)|$, we conclude that for sufficiently small h there exists a constant C with

$$(\|u\| + |u(1)|) \leq \frac{C}{k}.$$

It then follows that

$$\forall v \in V_h: \quad |\mathcal{B}(z,v)| \leq \frac{C}{k}|v|_1$$

and the proof is completed.

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