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THE PARTITION OF UNITY METHOD

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SUMMARY

A new finite element method is presented that features the ability to include in the finite element space knowledge about the partial differential equation being solved. This new method can therefore be more efficient than the usual finite element methods. An additional feature of the partition-of-unity method is that finite element spaces of any desired regularity can be constructed very easily. This paper includes a convergence proof of this method and illustrates its efficiency by an application to the Helmholtz equation for high wave numbers. The basic estimates for *a posteriori* error estimation for this new method are also proved.

KEY WORDS: finite element method; meshless finite element method; finite element methods for highly oscillatory solutions

1. INTRODUCTION

We present a new method, the Partition of Unity Method (PUM), which allows for the construction of conforming ansatz spaces with local properties determined by the user. The development of this method was motivated by the need for new techniques for the solution of problems where the classical FEM approaches fail or are prohibitively expensive; for example, equations with rough coefficients (arising e.g. in the modelling of composites, materials with microstructure, stiffeners, etc.) and problems with boundary layers or highly oscillatory solutions fall into that category. The approach taken in the PUM is to start from a variational formulation and then design the trial (and test) spaces in view of the problem under consideration.
The main features of the PUM are the following.

1. The PUM permits inclusion of *a priori* knowledge about the differential equation in the ansatz spaces.
2. The PUM allows us to construct easily ansatz spaces of any desired regularity; therefore, trial spaces for the use in variational formulations of higher-order differential equations (e.g. various plate and shell models) are available.

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Let us elaborate these two features in more detail as they represent the core ideas of the PUM: The first point is related to the local approximation properties of the spaces constructed by the PUM, whereas the second point is intimately linked to the conformity of these spaces.

A necessary condition for an ansatz space to perform well is that it can approximate the exact solution well locally. In the classical FEM, this local approximation is achieved by (mapped) polynomials. However, if analytic knowledge about the local behaviour of the exact solution is available, local approximation may be done with functions better suited than polynomials. Therefore, in order to illustrate the first feature of the PUM, let us give a few examples of problems where analytic knowledge about the local behaviour of the solution is available. The simplest example is provided by Laplace's equations, $-\Delta u = 0$, in two dimensions. Locally, the solutions of Laplace's equation can be approximated by harmonic polynomials (i.e. polynomials which satisfy Laplace's equation) of degree p, and this approximation is essentially as good as the approximation with all polynomials of degree p (see Section 7.1). However, there are only $2p + 1$ (linearly independent) harmonic polynomials of degree p, whereas the dimension of the full spaces of polynomials of degree p is $O(p^2)$. This simple example extends to elliptic equations with analytic coefficients. There too, it is possible to define 'generalized harmonic polynomials' which have approximation properties similar to those of the harmonic polynomials.[1–5]

For Laplace's equation or the elasticity equations, corners or sudden changes of boundary conditions introduce certain types of singularities. A precise description of these singularities can be used to construct very efficient non-polynomial shape functions to deal with these singularities[6,7] (see also Section 5.1).

The classical FEM relies (among other things) on the local approximation properties of polynomials. For certain types of problems, such as equations with rough coefficients or problems with highly oscillatory solution, polynomials have poor approximation properties. As is shown by Babuška *et al.*,[8] the choice of non-polynomial ansatz functions which are custom tailored to the problem leads to optimal rates of convergence, whereas the classical FEM, relying on the approximation properties of polynomials, perform extremely poorly. A similar situation arises in the approximation of highly oscillatory functions, e.g. the solutions of Helmholtz's equation. It was demonstrated by Melenk[5] that the approximation with plane waves displaying the same oscillatory behaviour as the solution can be very efficient.

Finally, another example where non-polynomial approximation spaces are of interest is furnished by problems on unbounded domains. For problems such as Laplace's equation or the Helmholtz equation, expansions of the solution around the point at infinity are known, and it may be desirable to build ansatz spaces based on these expansions. The PUM provides the framework to do so (cf. also References 8–12 and references therein for the use of so-called infinite elements in a more conventional setting).

Let us now turn to the second feature of the PUM, the ability to construct ansatz spaces of any desired regularity. In order to see this, we need to describe briefly how the PUM works—a more detailed description can be found in Section 3. Let overlapping patches $\{\Omega_i\}$ be given which comprise a cover of the domain Ω, and let $\{\varphi_i\}$ be a partition of unity subordinate to the cover. On each patch, let function spaces V_i reflect the local approximability. Then the global ansatz space V is given by $V = \sum_i \varphi_i V_i$. Local approximation in the spaces V_i can be either achieved by the smallness of the patches (an 'h version') or by good properties of V_i (a 'p version'). Theorem 1 below states that the global space V inherits the approximation properties of the local spaces V_i. Additionally, it inherits the smoothness of the partition of unity (and the spaces V_i). In particular, the global space V can be made to be conforming by choosing the partition of unity judiciously. Moreover, the construction of smoother ansatz spaces, which are necessary for the

use in variational formulations of various plate and shell models, is easily possible by using a partition of unity which is sufficiently smooth.

For a successful implementation of the PUM, three issues have to be addressed. We will merely state them here and note that they are intrinsically similar to the implementational issues of the meshless methods discussed in Section 2.3.

1. The integration of the shape functions constructed by the PUM.
2. Finding a basis of the PUM space and controlling the condition number of the stiffness matrix created by the PUM.
3. The implementation of essential boundary conditions.

The paper is organized as follows. In Section 2 we discuss the relation of the PUM to other methods, namely, the *h–p* version of the FEM and the various meshless methods proposed recently. In Section 3 we develop the PUM and give a proof of its approximation properties. In Sections 4.1–4.4, we illustrate some of the features of the PUM in a one dimensional setting. In Section 4.1, we demonstrate how the PUM produces robust finite element spaces for a problem with a boundary layer. The performance of the PUM for this particular problem is comparable to the usual finite element methods for problems with smooth solutions because the PUM allows us to create finite element spaces which capture precisely the behaviour of the boundary layer. Section 4.2 proposes several types of partitions of unity which satisfy the necessary conditions for the PUM to work. Section 4.3 analyses in more detail the case of polynomial local approximation spaces. In particular, the problem of potential linear dependencies and the issue of the condition number of the stiffness matrix is addressed. In Section 4.4 finally, a PUM is exhibited in which all the degrees of freedom have the meaning of the value of the approximating function in appropriate points. Sections 5.1 and 5.2 discuss briefly methods how to choose good local approximation spaces and the issue of the optimality of local spaces. Two numerical examples are presented. In Section 7.1, the PUM is compared with the usual *p* versions for the approximation of harmonic functions. In Section 7.2 the PUM is used for the approximation of solutions to Helmholtz's equation with large wave number. The PUM is shown to be superior (both in terms of error per degree of freedom and error per floating point operation) to several *h* version type finite element methods. The paper concludes in Section 8 with a proof of an *a posteriori* error estimator for the PUM which is based on exact solutions of appropriate local problems.

2. THE RELATION OF THE PUM TO OTHER METHODS

2.1. The classical h, p, and h–p methods

In this Section we argue that the PUM can be viewed as a generalization of the classical *h* and *p* version and that the performance of the standard methods also hinges on the availability of good local descriptions of the solution. This close connection between the ability to describe the local behaviour of the solution and the performance of the method was our main theme in the discussion of the first feature of the PUM in Section 1.

Let us first recall the main approximation properties of the classical FEM spaces. The trial spaces of the classical *h*, *p*, and *h–p* methods are spaces of piecewise polynomials which are continuous across inter-element boundaries. The key point for the approximation properties of these classical ansatz spaces is that locally (here: on each element) approximation is done by polynomials and that these local approximations may satisfy the additional constraint to be conforming (here: they satisfy an inter-element continuity condition) without sacrificing the approximation properties. In the *h* version, the polynomial degree *p* is fixed (typically, $p \leqslant 2$) and

approximation is achieved by decreasing the mesh size h. An appropriate interpolant (e.g. for $p = 1$ on triangles, the nodal interpolant can be taken) produces a good approximation which satisfies the necessary continuity requirements. In the p version, local approximation is realized by polynomials of increasingly higher degree. The approximation properties of conforming p extensions are due to two facts. Unconstrained, i.e. without any inter-element continuity constraints, polynomials have good approximation properties on each patch. The resulting jumps across inter-element boundaries can be resolved by polynomial corrections because polynomial spaces are—in contrast to, e.g. spaces of harmonic polynomials—large enough to admit continuous extensions from the element boundaries into the elements.[13]

The PUM can be viewed as a generalization of the classical h and p versions if the local approximation spaces V_i in the PUM are chosen to be polynomial spaces. The approximation properties of the PUM spaces constructed in this way are very similar to those of the classical spaces; see Examples 1 and 2 in Section 3 for the details. The mesh design (i.e. determining areas of mesh refinement) and the choice of the polynomial degree on each element affect greatly the performance of the h–p FEM. The design of the mesh and the p degree distribution depend on knowledge about the local behaviour of the solution. For example, in elliptic problems with piecewise analytic input data, a very good description of the local behaviour of the solution is available in terms of so-called countably normed spaces.[14] This precise knowledge about the local behaviour of the solution allows us to control the mesh refinement towards the singularities and the polynomial degree distribution in an optimal way, and we obtain exponential rates of convergence (cf. Reference 15 for an overview of state of the art h–p technology).

2.2. *Data fitting*

We include a brief section on data fitting because data fitting is also concerned with approximation problems and therefore ideas from data fitting can be and are used in the design of ansatz spaces. We will merely outline some of the ideas used in data fitting and refer to References 16 and 17 for more complete surveys on the subject. Typically in data fitting the fitting algorithm produces a function F which is of the form $F(x) = \sum_i f_i \Phi_i(x)$ where the values f_i are the data (e.g. function values, derivatives, etc.). We will call the functions Φ_i the shape functions.

One class of fitting algorithms is based on the so-called 'inverse distance weighted methods' whose ancestor is Shephard's[18] method. In the basic Shephard method, scattered data (x_i, f_i) is interpolated by a function

$$F(x) = \sum_i f_i w_i(x) \Big/ \sum_i w_i(x)$$

where the weights are typically chosen as decaying functions of the distance of x from the points x_i. The basic shape functions are thus

$$\Phi_i(x) = \frac{w_i(x)}{\sum_j w_j(x)}$$

Note that these functions form a partition of unity. Furthermore, it is worth noting that the approximation properties of the interpolant F can be understood by interpreting $\sum_i f_i \Phi_i$ as a suitable approximation of an integral of the form $\int f(y)\Phi_x(y)\,\mathrm{d}y$ where Φ_x is an approximation

of the Delta function concentrated at the point x. One possible generalization[19] of the classical Shephard method is to seek the interpolating function F in the form

$$F(x) = \tilde{F}(x, a_0, \ldots, a_p)$$

where the parameters $a_0, \ldots, a_p$ are determined by minimizing (for each fixed x; therefore, strictly speaking, the coefficients $a_0, \ldots, a_p$ depend on x)

$$\sum_i [f_i - \tilde{F}(x_i, a_0, \ldots, a_p)]^2 \, w_i(x) \tag{1}$$

If the function $\tilde{F}$ is a polynomial in x and the parameters are the coefficients of the polynomial, this method is called moving least-squares method.[20] If one carries out this minimization, the global approximation F takes the form

$$F(x) = \sum_i \tilde{\Phi}_i(x) f_i$$

and the desired shape functions are the functions $\tilde{\Phi}_i$. Note that the evaluation of the shape functions $\tilde{\Phi}_i$ at each point involves the solution of a minimization problem with $p + 1$ unknowns; note furthermore that some conditions have to be satisfied by the relative position of the nodal points x_i in order to guarantee a unique minimizer. This method is a generalization of Shephard's method because the particular choice of $\tilde{F}(x, a_0) = a_0$, i.e. polynomials of degree 0, reduces the method to Shephard's method.

A different generalization of the classical Shephard method uses the fact that the functions Φ_i form a partition of unity. For example, if the function values and the derivatives are given in the nodes x_i, then one can choose the function

$$F(x) = \sum_i \Phi_i(x) \, L_i(x) \tag{2}$$

where the L_i are the Taylor polynomials about the nodes x_i. In fact, any local approximation may do. This is the starting point of the PUM.

Besides these 'inverse distance weighted methods' there are many other ideas used in data fitting. For example, a popular technique is to use the classical finite element shape functions, viz., use piecewise polynomial interpolation. Another set of methods is based on the so-called radial basis functions (see Reference 17 for an overview), also known as multiquadrics. We would like to emphasize here that these radial basis function methods are, in their original form, non-local and therefore do not lead to sparse stiffness matrices (radial basis functions with compact support have been constructed recently, cf. the survey in Reference 17).

We will see in Section 2.3 that the shape functions used in data fitting can be used in various variational formulations for the solution of partial differential equations. However, one should notice that the shape functions used in data fitting are designed to have certain features which may not be very important in Galerkin methods. For example, many data-fitting shape functions are smooth (C^1 or C^2, say) because they were designed to produce good results in visualization applications. In contrast to this, smoothness of the ansatz functions in Galerkin methods is dictated by the choice of the variational formulation and typically much less regularity is required than the data-fitting shape functions provide.

Another observation regarding the use of data fitting shape functions in Galerkin methods is the fact that the goal of data fitting is to interpolate a given data set. However, in Galerkin methods it is enough that the shape functions have good approximation properties—approximation rather than interpolation is important. Therefore, the stability issues which play an

important role in the design of data fitting shape functions should not be as prominent in the design of shape functions to be used in Galerkin methods.

In accordance with our discussion of the first feature of the PUM of Section 1, we also note that the data fitting algorithms do not take into account the fact that we are trying to solve a partial differential equation. As a matter of fact, the 'inverse distance weighted methods' described above rely on local polynomial fitting. As we mentioned above, the approximation properties of the various data fitting schemes discussed in this section can be understood by interpreting the interpolant F as coming from a convolution with an approximate Delta function. For historical completeness' sake, let us point out that the use of approximate Delta functions, or weighted averages, to obtain (smooth) approximations of a given function has been a standard tool in mathematical analysis for a long time. To give just a few examples, let us mention the proof of the density of smooth functions in Sobolev spaces, the proof of the density of polynomials in C^0 by means of Bernstein's polynomials, and various results on the convergence of Fourier series.

2.3. *Meshless methods*

The basic idea of meshless methods is to use shape functions which are used in data fitting as the ansatz functions in a variational formulation. More precisely, given a distribution of nodes x_i, the fitting algorithm is invoked to produce the shape functions Φ_i which are then in turn used as the ansatz function of a Galerkin or collocation method. This is the essential idea of the DEM,[21] the EFGM,[22–24] the *hp*-clouds,[25] multiquadric methods,[26,27] the SPH,[28] the RKPM,[29–31] and the GFDM.[32] We refer to Reference 33 for a detailed comparison and a study of the connections between these various methods.

In the SPH and the RKPM, for example, the following 'interpolation scheme' motivates the choice of the ansatz functions. Starting from a suitable approximation of the Delta function ω_x, concentrated at the point x, the integral $f(x) \approx \int f(y)\omega_x(y)\,\mathrm{d}y$ is suitably approximated in order to produce an approximation of the form $\sum_i f(y_i)\tilde{w}_i(x)$. The functions $\tilde{w}_i$ are then taken as the shape functions for a collocation or Galerkin method. Of course, there is great freedom in the choice of the function ω_x. In the RKPM this freedom is exploited to impose additional conditions, in particular that polynomials of a fixed degree are reproduced.

Using equation (1) with $\tilde{F}$ as polynomials in x leads to the so-called moving least-squares method and the shape functions produced by that fitting algorithm are the ones used by the DEM and the EFGM in a Galerkin method. The PUM could be viewed as a generalization of the idea presented in equation (2). Instead of trying to approximate locally with Taylor polynomials, however, we permit any good local approximation L_i. The *hp* clouds can be viewed as a mixture of the EFGM and the PUM if the local approximation is realized by polynomials. Actually, in its simplest form, the *hp* clouds are a Galerkin method whose shape functions are given by equation (2) where the L_i are polynomials or degree p.

Let us finally highlight the differences between the PUM and the various meshless methods discussed above. The shape functions of all the above meshless methods are motivated by some form of local *polynomial* fitting. Therefore, the approximation properties of the ansatz spaces used by these methods are essentially given by the local approximation properties of polynomials and all these methods are very similar to the PUM if the local approximation spaces in the PUM are chosen to be polynomials. In fact, the computational analysis of Oden and Duarte[25] indicates that in this case the PUM is more efficient than the EFGM. Since the performance (from an approximation point of view) of these meshless methods rests on the local approximation properties of polynomials, we may expect these methods to perform poorly whenever polynomials approximate the solution poorly, as is the case in problems with rough coefficients or the

problems with highly oscillatory solutions alluded to in Section 1. Note that these are precisely the situations where the classical FEM perform poorly as well. In contrast to this, the PUM has much greater flexibility in the choice of the local approximation and allows us to approximate locally with functions which are custom tailored to the problem under consideration. Therefore, we may hope that the PUM can deal successfully with problems with which the classical FEM or these meshless methods cannot cope.

3. MATHEMATICAL FOUNDATIONS OF THE PUM

In this section, we present our method of constructing conforming subspaces of $H^1(\Omega)$. We construct ansatz spaces which are subspaces of $H^1(\Omega)$ as an example because of their importance in applications. We would like to stress that the method leads to the construction of smoother spaces in a straightforward manner. Crucial to the construction of the PUM spaces is the notion of a (M, C_∞, C_G) partition of unity.

Definition 1. Let $\Omega \subset \mathbb{R}^n$ be an open set, $\{\Omega_i\}$ be an open cover of Ω satisfying a pointwise overlap condition

$$\exists M \in \mathbb{N} \quad \forall x \in \Omega \quad \text{card}\{i \,|\, x \in \Omega_i\} \leqslant M$$

Let $\{\varphi_i\}$ be a Lipschitz partition of unity subordinate to the cover $\{\Omega_i\}$ satisfying

$$\text{supp } \varphi_i \subset \text{closure}(\Omega_i) \quad \forall i \tag{3}$$

$$\sum_i \varphi_i \equiv 1 \quad \text{on } \Omega \tag{4}$$

$$\|\varphi_i\|_{L^\infty(\mathbb{R}^n)} \leqslant C_\infty \tag{5}$$

$$\|\nabla\varphi_i\|_{L^\infty(\mathbb{R}^n)} \leqslant \frac{C_G}{\text{diam}\,\Omega_i} \tag{6}$$

where C_∞, C_G are two constants. Then $\{\varphi_i\}$ is called a (M, C_∞, C_G) partition of unity subordinate to the cover $\{\Omega_i\}$. The partition of unity $\{\varphi_i\}$ is said to be of *degree* $m \in \mathbb{N}_0$ if $\{\varphi_i\} \subset C^m(\mathbb{R}^n)$. The covering sets $\{\Omega_i\}$ are called *patches.*

Definition 2. Let $\{\Omega_i\}$ be an open cover of $\Omega \subset \mathbb{R}^n$ and let $\{\varphi_i\}$ be a (M, C_∞, C_G) partition of unity subordinate to $\{\Omega_i\}$. Let $V_i \subset H^1(\Omega_i \cap \Omega)$ be given. Then the space

$$V := \sum_i \varphi_i V_i = \left\{ \sum_i \varphi_i v_i \,|\, v_i \in V_i \right\} \subset H^1(\Omega)$$

is called the *PUM space*. The PUM space V is said to be of degree $m \in \mathbb{N}$ if $V \subset C^m(\Omega)$.The spaces V_i are referred to as the *local approximation spaces.*

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be given. Let $\{\Omega_i\}$, $\{\varphi_i\}$, and $\{V_i\}$ be as in Definitions* 1 *and* 2. *Let $u \in H^1(\Omega)$ be the function to be approximated. Assume that the local approximation spaces V_i have the following approximation properties*: *On each patch $\Omega_i \cap \Omega$, u can be approximated by a function $v_i \in V_i$ such that*

$$\|u - v_i\|_{L^2(\Omega_i \cap \Omega)} \leqslant \varepsilon_1(i)$$

$$\|\nabla(u - v_i)\|_{L^2(\Omega_i \cap \Omega)} \leqslant \varepsilon_2(i)$$

Then the function

$$u_{\mathrm{ap}} = \sum_i \varphi_i v_i \in V \subset H^1(\Omega)$$

satisfies

$$\|u - u_{\mathrm{ap}}\|_{L^2(\Omega)} \leqslant \sqrt{M}\, C_\infty \left(\sum_i \varepsilon_1^2(i) \right)^{1/2}$$

$$\|\nabla(u - u_{\mathrm{ap}})\|_{L^2(\Omega)} \leqslant \sqrt{2M} \left(\sum_i \left(\frac{C_G}{\operatorname{diam} \Omega_i} \right)^2 \varepsilon_1^2(i) + C_\infty^2 \varepsilon_2^2(i) \right)^{1/2}$$

Proof. Using the fact that $\sum_i \varphi_i \equiv 1$ on Ω, we can write $u - u_{\mathrm{ap}} = \sum_i \varphi_i (u - v_i)$. The theorem follows after an application of the second estimate of Lemma 2 (see Section 8) with $u_i = \varphi_i (u - v_i)$. □

Example 1. The PUM as an h version. Let $u \in H^k(\Omega)$, $k \geqslant 1$. Let each patch Ω_i have diameter $h_i \leqslant h$, and let each V_i have approximation properties

$$\begin{aligned} \varepsilon_1(i) &\leqslant C h_i^{\mu+1} \|u\|_{H^k(\Omega \cap \Omega_i)} \\ \varepsilon_2(i) &\leqslant C h_i^{\mu} \|u\|_{H^k(\Omega \cap \Omega_i)} \end{aligned} \tag{7}$$

for some appropriate $\mu > 0$. Then the error estimates of Theorem 1 take the form

$$\begin{aligned} \|u - u_{\mathrm{ap}}\|_{L^2(\Omega)} &\leqslant M C_\infty C h^{\mu+1} \|u\|_{H^k(\Omega)} \\ \|\nabla(u - u_{\mathrm{ap}})\|_{L^2(\Omega)} &\leqslant M C \sqrt{2(C_G + C_\infty)}\, h^{\mu} \|u\|_{H^k(\Omega)} \end{aligned} \tag{8}$$

where we used the first estimate of Lemma 2 (Section 8) in the estimate of the sums $\sum_i \varepsilon_1(i)^2$, $\sum_i \varepsilon_2(i)^2$. Note that estimate (8) holds for any system of local approximation spaces V_i satisfying (7). For example, if the spaces V_i consist of polynomials of degree p, then (7) holds with $\mu = \min(k-1, p)$. If the spaces V_i consist of harmonic polynomials of degree p, (7) holds also with $\mu = \min(k-1, p)$ if we know *a priori* that the function u is harmonic. In this example, local approximability of the spaces V_i (and thus global approximability by Theorem 1) is achieved by the smallness of the patches $\Omega_i \cap \Omega$.

Example 2. The PUM as a p version. Let $u \in H^k(\Omega)$, $k \geqslant 1$, and let $\{\Omega_i\}_{i=1}^N$ be N fixed patches covering Ω. Denote $\operatorname{diam}(\Omega_i)$ by h_i. Assume that the spaces V_i (depending on a parameter p) have the approximation properties

$$\begin{aligned} \varepsilon_1(i) &\leqslant C h_i p^{-\mu} \|u\|_{H^k(\Omega \cap \Omega_i)} \\ \varepsilon_2(i) &\leqslant C p^{-\mu} \|u\|_{H^k(\Omega \cap \Omega_i)} \end{aligned} \tag{9}$$

for some appropriate $\mu > 0$. Then the error estimates of Theorem 1 take the form

$$\|u - u_{\mathrm{ap}}\|_{L^2(\Omega)} \leqslant M C_\infty C \max_i h_i p^{-\mu} \|u\|_{H^k(\Omega)},$$

$$\|\nabla(u - u_{\mathrm{ap}})\|_{L^2(\Omega)} \leqslant M C \sqrt{2(C_{\mathrm{G}}^2 + C_\infty^2)}\, p^{-\mu} \|u\|_{H^k(\Omega)}$$

Note that this estimate holds for any system V_i satisfying (9)—they do not have to polynomials of degree p. If the spaces V_i consist of polynomials of degree p then (9) holds with $\mu = k - 1$ (under

some appropriate conditions on the shape of the patches). Estimate (9) also holds for spaces V_i consisting of harmonic polynomials of degree p if the function u is known to be harmonic (see Theorems 2 and 3). In this example, the approximation properties of the global PUM space are achieved through increased approximability of the local spaces while keeping the patches fixed. If we allow the size of the patches to vary as well, then this method behaves like an *hp* version.

We would like to stress at this point that the requirements on the partition of unity are very weak: it only needs to be Lipschitzian in order to produce H^1 subspaces. Also, we do not need positivity of the partition of unity—the elements of the partition of unity are allowed to change sign. Moreover, if the partition of unity is of degree m (and the local approximation spaces are sufficiently smooth), then the finite element space V as constructed in Definition 2 is also of degree m.

Theorem 1 is formulated in terms of H^1, appropriate for a large class of second-order problems. *Mutatis mutandis*, however, the estimates can be formulated in terms of H^k, $k \geqslant 1$ to produce ansatz spaces for higher-order equations. Similar estimates can be achieved in Sobolev spaces $W^{k,p}$.

Remark 1. This idea of using a partition of unity to construct finite element spaces tailored to the differential equation has been used by Babuška *et al.*[8] and Melenk[34,5]. As mentioned in Section 2.3, for a judicious choice of parameters, the EFGM reduces to a special type of PUM, and the convergence analysis of Oden and Duarte[25] for this special case is based on Theorem 1.

4. THE PUM IN ONE DIMENSION

4.1. A one dimensional example

Let us demonstrate for a one-dimensional model problem how ansatz spaces with good approximation properties are constructed with the PUM. To this end, consider

$$\begin{aligned} -u'' + k^2 u &= f \in C^2[0,1] \quad \text{on } (0,1) \\ u(0) &= 0 \\ u'(1) &= g \in \mathbb{R} \end{aligned} \tag{10}$$

We assume that the parameter $k > 1$ is large. Associated with this problem is an 'energy' norm, given by

$$\| v \|_{\mathrm{E}} := \{ \| v' \|^2_{L^2(\Omega)} + k^2 \| v \|^2_{L^2(\Omega)} \}^{1/2}.$$

Let us note that for large k, the solution to problem (10) typically exhibits a boundary layer in the neighbourhood of $x = 0$ (and a weaker boundary layer at $x = 1$), and thus the usual FEM performs poorly unless h is sufficiently small (relative to k^{-1}) or a very strongly refined mesh is used. The PUM allows us to use local spaces reflecting this behaviour, and therefore leads us to a robust FEM, i.e. a method which is good uniformly in k.

Let $n \in \mathbb{N}$, $h = 1/n$ and define $x_j = jh$, $j = 0, \ldots, n$. Define also $x_{-1} = -h$, $x_{n+1} = 1 + h$ and set the patches $\Omega_j = (x_{j-1}, x_{j+1})$, $j = 0, \ldots, n$. On each patch Ω_j, we have to define a local space which can approximate the solution u of problem (10) well. We consider

$$V_j^{\mathrm{I}} = \operatorname{span}\{1, \sinh kx, \cosh kx\} \quad \text{on } \Omega_j \cap \Omega, \quad j = 1, \ldots, n$$

$$V_0^{\mathrm{I}} = \operatorname{span}\{\sinh kx, 1 - \cosh kx\} \text{ on } \Omega_0 \cap \Omega$$

We note that the space V_0 is constructed such that it satisfies the essential boundary condition at $x = 0$. The approximation properties of these spaces, which are tailored to this particular problem (10), are given by the following lemma.

Lemma 1. Let *u be the solution to problem* (10) *and let Ω_j, V_j^{I} be as defined above. Then there are $v_j \in V_j^{\mathrm{I}}$ such that*

$$\|(u - v_j)'\|_{L^2(\Omega_j \cap \Omega)} \leqslant Ch^{1/2} \left[h^2 \min(1, (kh)^{-2}) \|f'\|_{L^\infty(\Omega)} + \frac{h^2}{k} \|f''\|_{L^\infty(\Omega)} \right]$$

$$\|(u - v_j)\|_{L^2(\Omega_j \cap \Omega)} \leqslant Ch^{1/2} \left[h^3 \min(1, (kh)^{-2}) \|f'\|_{L^\infty(\Omega)} + \frac{h^2}{k} \min(h, k^{-1}) \|f''\|_{L^\infty(\Omega)} \right]$$

where $C > 0$ is independent of h, k, and f.

Proof. Because the spaces V_j^{I} contain the fundamental system $\{\sinh kx, \cosh kx\}$, it is enough to approximate a particular solution to

$$-u'' + k^2 u = f \quad \text{on } \Omega_j \cap \Omega$$

By Taylor's theorem, on $\Omega_j \cap \Omega$, $f(x) = l(x) + r(x)$ where $l(x)$ is linear and $|r(x)| \leqslant (2h)^2 \|f''\|_{L^\infty(\Omega)}$ (note that $\operatorname{diam} \Omega_j \leqslant 2h$). A particular solution to the problem with the right-hand side $r(x)$ is given by the solution u_r to

$$-u_r'' + k^2 u_r = r \quad \text{on } \Omega_j \cap \Omega$$
$$u = 0 \quad \text{on } \partial(\Omega_j \cap \Omega)$$

Thus,

$$\|u_r'\|^2_{L^2(\Omega_j \cap \Omega)} + k^2 \|u_r\|^2_{L^2(\Omega_j \cap \Omega)} \leqslant \frac{2h}{k^2} \|r\|^2_{L^\infty(\Omega_j \cap \Omega)}$$

from whence

$$\|u_r'\|_{L^2(\Omega_j \cap \Omega)} \leqslant Ch^{1/2} \frac{h^2}{k} \|f''\|_{L^\infty(\Omega)}$$

$$\|u_r\|_{L^2(\Omega_j \cap \Omega)} \leqslant Ch^{1/2} \frac{h^2}{k} \min(h, k^{-1}) \|f''\|_{L^\infty(\Omega)}$$

with $C > 0$ independent of h, k, and f. Finally, a particular solution to the problem with the right-hand side $l(x)$ is given by $u_l(x) = k^{-2} l(x)$ which can be approximated in V_j^{I} such that

$$\|u_l - v_j\|_{L^2(\Omega_j \cap \Omega)} + h \|(u_l - v_j)'\|_{L^2(\Omega_j \cap \Omega)} \leqslant Ch^3 h^{1/2} \min(1, (kh)^{-2}) \|f'\|_{L^\infty(\Omega)},$$

where $C > 0$ is independent of h, k, and f. The assertion of the lemma follows. □

Remark 2. The spaces V_j^{I} were chosen as local approximation spaces because they contain the fundamental system $\{\sinh kx, \cosh kx\}$ and the particular solution that corresponds to a constant right-hand side. It is easy to check that the functions $\{1, x, \ldots, x^p\}$ actually span a space of particular solutions for polynomial right-hand sides of degree p. Hence, Lemma 1 can be adapted to produce the following approximation result. The spaces

$$V_j^{\mathrm{I}} = \operatorname{span}\{\sinh kx, \cosh kx, 1, x, \ldots, x^p\} \quad \text{on } \Omega_j \cap \Omega, \quad j = 1, \ldots, n$$
$$V_0^{\mathrm{I}} = \operatorname{span}\{\sinh kx, 1 - \cosh kx, x, \ldots, x^p\} \quad \text{on } \Omega_0 \cap \Omega$$

contain $v_j \in V_j^{\mathrm{I}}$ such that

$$\|(u - v_j)'\|_{L^2(\Omega_j \cap \Omega)} \leqslant C_p h^{1/2} \left[h^{p+2} \min\left(1, (kh)^{-2}\right) \|f^{(p+1)}\|_{L^\infty(\Omega)} + \frac{h^{p+2}}{k} \|f^{(p+2)}\|_{L^\infty(\Omega)} \right]$$

$$\|(u - v_j)\|_{L^2(\Omega_j \cap \Omega)} \leqslant C_p h^{1/2} \left[h^{p+3} \min(1, (kh)^{-2}) \|f^{(p+1)}\|_{L^\infty(\Omega)} + \frac{h^{p+2}}{k} \min(h, \tfrac{1}{k}) \|f^{(p+2)}\|_{L^\infty(\Omega)} \right]$$

for some C_p independent of h, k, p, and f.

For any partition of unity $\{\varphi_j\}$ subordinate to the covering $\{\Omega_j\}$, the finite element space V^{I} as constructed in Definition 2 is given by

$$V^{\mathrm{I}} = \operatorname{span}\{\varphi_j(x),\ \varphi_j(x)\sinh kx,\ \varphi_j(x)\cosh kx,$$
$$\varphi_0(x)\sinh kx,\ \varphi_0(x)(1 - \cosh kx) \mid j = 1, \ldots, n\}.$$

Since the assumptions on the partition of unity stipulate that the functions φ_j be Lipschitz continuous, we see that $V^{\mathrm{I}} \subset H^1(\Omega)$. Because each function φ_j is assumed to vanish outside the patch Ω_j, and because the elements of V_0^{I} vanish at $x = 0$, we see that all elements of V^{I} vanish at $x = 0$. Hence, a conforming finite element method can be based on V^{I}, and the finite element solution is the best approximant in the energy norm

$$\|u - u_{\mathrm{FE}}\|_{\mathrm{E}} \leqslant \inf_{v \in V^{\mathrm{I}}} \|u - v\|_{\mathrm{E}}.$$

Therefore, with the aid of Theorem 1, the local approximation properties of the spaces V_j^{I} in Lemma 1 lead to

Proposition 1. *Let the patches $\{\Omega_j\}$ and the local approximation spaces $\{V_j^{\mathrm{I}}\}$ be given as above. Let $\{\varphi_j\}$ be a $(M, C_\infty, C_{\mathrm{G}})$ partition of unity subordinate to the patches $\{\Omega_j\}$. Then the finite element solution u_{FE} of the PUM satisfies*

$$\|u - u_{\mathrm{FE}}\|_{\mathrm{E}} \leqslant Ch^2 \{\min(1, (kh)^{-1}) \|f'\|_{L^\infty(\Omega)} + k^{-1} \|f''\|_{L^\infty(\Omega)}\} \tag{11}$$

where $C > 0$ is independent of h, k, and f.

This shows that the PUM enables us to construct robust finite element methods which are efficient uniformly in k, i.e. the finite element method behaves as well for the rough case of large k as it does for the smooth case $k = 1$. The PUM gives these good uniform estimates because the local spaces V_j^{I} capture the local behaviour of the exact solution very well. Note that the number of degrees of freedom is comparable to the number of degrees of freedom of the usual, piecewise quadratic finite element method which is — with the exception of (piecewise) quadratic solutions — of order h^2 and not better. Thus, the PUM is as good as the usual piecewise quadratic finite element method for the smooth case $k = 1$.

A simple adaptation of this idea is to choose the local spaces selectively. For example, since the right-hand side f is smooth, we expect a boundary layer close to $x = 0$, $x = 1$, but expect smooth behaviour away from the boundary. Hence, it suffices to use the spaces V_j^{I} on patches close to the points $x = 0$, $x = 1$, and we can use polynomials spaces $V_j = \operatorname{span}\{1, x, \ldots, x^p\}$ on patches away from the boundary. The idea of choosing the local approximation spaces selectively can also be employed in adaptive versions of the PUM. Keeping the patches and changing the degree p of the polynomials lets the PUM act like an adaptive p version; changing the size of the patches adaptively makes the PUM behave like an adaptive h version.

Remark 3. The idea of using non-polynomial ansatz functions in order to capture the behaviour of the exact solution has been used for a long time. For example, L–splines contain the fundamental system of the equation under consideration. We elaborated this one-dimensional example of the PUM because it exhibits the main features of the PUM and most of the observations of the one-dimensional case carry over to the two- or three-dimensional setting.

4.2. Examples of partitions of unity

In this section we propose several (M, C_∞, C_G) partitions of unity for the one-dimensional example of the preceding section. Thus, the underlying cover of the domain (0, 1) is the one given in the previous section.

The usual piecewise linear hat functions form a partition of unity. Let

$$\varphi^1(x) = \begin{cases} 1 + \dfrac{x}{h} & \text{for } x \in (-h, 0] \\ 1 - \dfrac{x}{h} & \text{for } x \in (0, h) \\ 0 & \text{elsewhere} \end{cases} \tag{12}$$

and define the partition of unity by $\varphi_j^1(x) = \varphi(x - x_j)$, $j = 0, \ldots, n$.

2. Functions which are identically 1 on a subset of their support can also form a partition of unity:

$$\varphi^2(x) = \begin{cases} \dfrac{3}{2} + 2\dfrac{x}{h} & \text{for } x \in \left(-\dfrac{3}{4}h, -\dfrac{h}{4} \right] \\ 1 & \text{for } x \in \left(-\dfrac{h}{4}, \dfrac{h}{4} \right] \\ \dfrac{3}{2} - 2\dfrac{x}{h} & \text{for } x \in \left(\dfrac{h}{4}, \dfrac{3}{4}h \right) \\ 0 & \text{elsewhere} \end{cases} \tag{13}$$

and define the partition of unity by $\varphi_j^2(x) = \varphi(x - x_j)$, $j = 0, \ldots, n$.

3. A combination of the above two examples is to choose the functions φ_j^1 for patches in the interior but to modify the functions on patches close to the boundary. Define

$$\varphi^3(x) = \begin{cases} 1 + \dfrac{x}{h} & \text{for } x \in (-h, 0] \\ 1 & \text{for } x \in (0, h] \\ 2 - \dfrac{x}{h} & \text{for } x \in (h, 2h) \\ 0 & \text{elsewhere} \end{cases} \tag{14}$$

We observe that the patches $\Omega_0 \cup \Omega_1$, $\Omega_{n-1} \cup \Omega_n$ and Ω_j, $j = 2, \ldots, n-2$, cover Ω. On the patches Ω_j, $j = 2, \ldots, n-2$, we define $\varphi_j^3(x) = \varphi_j^1(x)$. On the patch $\Omega_0 \cup \Omega_1$ we choose $\varphi_1^3(x) = \varphi^3(x)$ and on the patch $\Omega_{n-1} \cup \Omega_n$ we choose $\varphi_{n-1}^3(x) = \varphi^3(x - x_{n-1})$. Note that $\varphi_1^3 = \varphi_0^1 + \varphi_1^1$ and $\varphi_{n-1}^3 = \varphi_{n-1}^1 + \varphi_n^1$.

4. In all three examples above, the partition of unity is merely Lipschitz continuous. However, partitions of unity of any desired regularity can be constructed. Here is a piecewise polynomial C^1 example. The resulting global finite element space V^{II} is then a subspace of C^1 [0,1]. Define

$$\varphi^4(x) = \frac{1}{h^3} \begin{cases} (x+h)^2(h-2x) & \text{for } x \in (-h, 0] \\ (h-x)^2(h+2x) & \text{for } x \in (0, h) \\ 0 & \text{elsewhere} \end{cases} \tag{15}$$

and define the individual members of the partition of unity by $\varphi_j^4(x) = \varphi^4(x - x_j)$ on the patches Ω_j.

5. In this example, let Ω_j be any cover of Ω satisfying an overlap condition (i.e. not more than M patches overlap in any given point $x \in \Omega$). Let ψ_j be Lipschitz continuous functions supported by the patches Ω_j. If $|\psi_j'| \leqslant C$ and $\sum_i \psi_i \geqslant \tilde{C}$ diam Ω_j on each $\Omega_j \cap \Omega$, for some C, $\tilde{C} > 0$ independent of j, then the functions

$$\varphi_j(x) = \frac{\psi_j(x)}{\sum_i \psi_i(x)}$$

form a $(M, C\tilde{C}^{-1}, C\tilde{C}^{-1}(1 + MC^2\tilde{C}^{-2}))$ partition of unity subordinate to the cover $\{\Omega_j\}$. Note that the functions ψ_j scale with their supports in the sense that $|\psi_j| \leqslant C\text{diam } \Omega_j$. The functions φ_j inherit the smoothness of the functions ψ_j, i.e. with this 'normalizing' technique, one can easily construct partitions of unity of any desired regularity. Another feature of the construction is that it allows us to build (M, C_∞, C_G) partitions of unity for very general covering situations. In particular, it enables us to produce the necessary partitions of unity whenever patches are added, removed or otherwise changed in an adaptive computational environment.

4.3. Polynomial local approximation spaces and linear dependencies

In this section we want to analyse in more detail the PUM spaces based on *polynomial* local approximation spaces. We will see below that for polynomial local approximation spaces, the choice of the partition of unity has an influence on the approximation properties of the PUM space and has implementational ramifications in the following sense. In any implementation, a basis of the PUM space has to be constructed, and it would be convenient if that basis were determined directly by the basis functions of the local approximation spaces. In general, however, this is not true. For example, for piecewise linear partitions of unity and polynomial local approximation spaces, the local basis functions (multiplied by the appropriate partition of unity functions) are linearly dependent and thus do not form a basis of the PUM space (see below). Although this example is artificial, it suggests that even if the local basis functions lead to a basis of the PUM space, the resulting functions might be 'nearly' linearly dependent, and the resulting finite element stiffness matrix will be badly conditioned.

Define

$$V_0^{\text{II}} = \text{span}\{x, \ldots, x^p\} \quad \text{on } \Omega_0 \cap \Omega$$

$$V_j^{\text{II}} = \text{span}\{1, x, \ldots, x^p\} = \text{span}\{1, x - x_j, \ldots, (x - x_j)^p\} \quad \text{on } \Omega_j \cap \Omega$$

for $j = 1, \ldots, n$ and set $V_0^{\mathrm{II}} = \{0\}$ if $p = 0$. For any partition of unity $\{\varphi_j\}$, the PUM space is given by

$$V^{\mathrm{II}} = \operatorname{span}\{\varphi_j(x)\, x^m, \varphi_0(x)\, x^q \,|\, j = 1, \ldots, n, \quad m = 0, \ldots, p, \quad q = 1, \ldots, p\}$$

The fact that the functions $\{\varphi_j\}$ form a partition of unity, i.e. $\sum_j \varphi_j(x) \equiv 1$ on Ω, implies that the space V^{II} satisfies a consistency condition in the sense that all polynomials of degree $\leqslant p$ which vanish in $x = 0$ are contained in V^{II}.

Let us now consider the spaces V^{II} based on the various partitions of unity of the previous section more closely. Denote by $V^{\mathrm{II},1}$, $V^{\mathrm{II},2}$, and $V^{\mathrm{II},3}$ the spaces V^{II} constructed using the partitions of unity $\{\varphi_j^1\}$, $\{\varphi_j^2\}$, and $\{\varphi_j^3\}$ respectively. Let us concentrate on $V^{\mathrm{II},1}$ first. Owing to the fact that the functions φ_j^1 are piecewise polynomials, the space $V^{\mathrm{II},1}$ is precisely the space of piecewise polynomials of degree $p+1$ constrained to vanish in $x = 0$. This is an example where the global finite element space has even better approximation properties than guaranteed by Theorem 1: Locally, approximation is done by polynomials of degree p and Theorem 1 states that the local approximation properties are inherited by the global space, i.e. the H^1 approximability is $O(h^p)$. However, the space of piecewise polynomials of degree $p+1$ has better approximation properties: it is $O(h^{p+1})$ for H^1 estimates. Let us note that $\dim V^{\mathrm{II},1} = n(p+1)$.

As mentioned above, it would be convenient for implementational purposes to take as a basis of the finite element space $V^{\mathrm{II},1}$ functions which are determined by the basis functions of the local spaces V_j^{II}, i.e. we would like to take the functions

$$\varphi_j^1(x)\,(x\text{-}x_j)^m, \quad j = 1, \ldots, n, \quad m = 0, \ldots, p \tag{16}$$

$$\varphi_0^1(x)^m, \quad m = 1, \ldots, p \tag{17}$$

However, these functions are not linearly independent for $p \geqslant 1$ as a simple counting argument reveals: there are $n(p+1)+p$ functions but $\dim V^{\mathrm{II},1} = n(p+1) < n(p+1)+p$ for $p \geqslant 1$. Of course, one can still use these functions. For problem (10) they will lead to a positive semi-definite matrix (as opposed to a positive definite matrix, which is obtained if a basis is used), which has many algebraic solutions. However, all these algebraic solutions are merely different representations of the same function on Ω. One way to solve this linear system is to use a penalty method to deal with the linear dependencies (see Reference 34 for a computational analysis).

One can avoid these linear dependencies if one uses a different partition of unity. For example, whenever the partition of unity is such that each member φ_j is identically 1 on an open set $\mathcal{O}_j \subset \Omega_j \cap \Omega$ (and all the other ones vanish there), linear dependencies as above cannot occur. Hence, the functions

$$\varphi_j^2(x)\,(x\text{-}x_j)^m, \quad j = 1, \ldots, n, \quad m = 0, \ldots, p \tag{18}$$

$$\varphi_0^2(x) x^m, \quad m = 1, \ldots, p \tag{19}$$

form indeed a basis of the space $V^{\mathrm{II},2}$.

A more careful analysis of the linear dependencies occurring for the case of $V^{\mathrm{II},1}$ reveals that the local approximation space at either the left or the right endpoint of Ω contains too many functions. Thus, a modification of the partition of unity at one (or both) endpoints allows us to exclude linear dependencies: The functions $\varphi_j^3(x)\,(x - x_j)^m, j = 2, \ldots, n-1, m = 0, \ldots, p$, $\varphi_1^3(x)$ x^m, $m = 1, \ldots, p$, form a basis of $V^{\mathrm{II},3}$. Let us point out that the space $V^{\mathrm{II},3}$ does no longer contain all piecewise polynomials of degree $p+1$. Let us note here that this space is very closely related to $V^{\mathrm{II},1}$. In fact, for problem (10), the stiffness matrix of the finite element method based on $V^{\mathrm{II},3}$ can be easily extracted from the positive semi-definite stiffness matrix constructed using $V^{\mathrm{II},1}$.

The example $V^{\mathrm{II},1}$ shows that 'unfortunate' combinations of local approximation spaces and partitions of unity exist, where the basis elements of the local spaces multiplied by the appropriate partition of unity function are linearly dependent. This indicates that even if the chosen functions derived from the local bases are linearly independent and form a basis of the finite element space, the resulting stiffness matrix may still be badly conditioned.

4.4. Polynomial local approximation spaces: Lagrange-type elements

If we choose the functions $\varphi_j(x)(x - x_j)^m$ as basis functions of the space V^{II}, the degrees of freedom cannot be identified directly as function values in certain points. Rather, the degrees of freedom are related to higher derivatives of the elements of V^{II} in the points x_j. In this sense, the functions $\varphi_j(x)(x - x_j)^m$ produce a Hermite type space. However, it is also possible to construct Lagrange type spaces, where the degrees of freedom represent the function values in particular 'Lagrange interpolation points'. Let us illustrate this for the case where we want to approximate locally with polynomials of degree p. Let $\{\Omega_j\}$ be a cover of $\Omega = (0, 1)$ and let $\{\varphi_j\}$ be a $(M, C_\infty, C_{\mathrm{G}})$ partition of unity subordinate to the cover. Let y_i, $i = 1, \ldots, N$, be the 'Lagrange interpolation points', and assume that there are $p + 1$ points y_i in each patch Ω_j. In order to be able to enforce the essential boundary condition at $x = 0$, we will stipulate $y_1 = 0$. On each patch Ω_j, let L_{j,y_i} be the usual polynomial Lagrange interpolation function of degree p which is 1 in the point y_i and vanishes in all the other p 'Lagrange interpolation points' which are in the patch Ω_j. As before, we define the global finite element space by

$$V^{\mathrm{III}} = \left\{ \sum_{i=1}^{N} \sum_{j} \varphi_j(x)\, L_{j,y_i}(x)\, a_{j,y_i} \,|\, a_{j,y_i} \in \mathbb{R} \right\}$$

This is exactly the same space as is obtained if the local spaces V_j are chosen to be $\mathrm{span}\{1, x, \ldots x^p\}$. Now, if we identify unknowns associated with the same interpolation point, i.e., if we set $a_{m,y_i} = a_{n,y_i}$ for all n, m for each point y_i, and denote these common values by a_{y_i}, we arrive at the space

$$V^{\mathrm{IV}} = \left\{ \sum_{i=1}^{N} \left[\sum_{j:y_i \in \Omega_j} \varphi_j(x)\, L_{j,y_i}(x) \right] a_{y_i} \,|\, a_{y_i} \in \mathbb{R} \right\}$$

Because the functions φ_j form a partition of unity and because the functions L_{j,y_i} take only the values 0 and 1 in the 'Lagrange interpolation points' y_m, the values a_{y_i} are precisely the function values of the elements of V^{IV}. Hence, we can take as a basis of V^{IV} the functions

$$\Phi_i(x) = \sum_{j:y_i \in \Omega_j} \varphi_j(x)\, L_{j,y_i}(x), \quad i = 1, \ldots, N$$

The essential boundary condition at $x = 0$ is also easily enforced by simply setting $a_{y_1} = 0$, which gives the space

$$V^{\mathrm{V}} = \left\{ \sum_{i=2}^{N} a_{y_i} \Phi_i(x) \,|\, a_{y_i} \in \mathbb{R} \right\}$$

Let us make a few remarks on the approximation properties of the space V^{V}. The approximation properties of the spaces V^{II} are given by the approximation properties of the local spaces V_j^{II}, i.e. by the approximation properties of polynomials of degree p. For fixed degree p and appropriate conditions on the distributions of the interpolation points on each patch, it can be shown that

approximation with V^{V} is — up to a constant — as good as with V^{II}. Finally, let us mention that these spaces V^{IV}, V^{V} are closely related to the EFGM.[22]

Above we noted that the close relation between the spaces $V^{II,1}$ and $V^{II,3}$ enables us to construct the stiffness matrix based on $V^{II,3}$ easily from the one based on $V^{II,1}$. Similarly, the stiffness matrix based on the functions Φ_i can be extracted from the stiffness matrix based on the functions $\varphi_j L_{j,y_i}$.

5. COMMENTS ON CHOOSING LOCAL APPROXIMATION SPACES

5.1. *Change of variables techniques*

In Section 4.1, we chose the local approximation spaces for problem (10) to consist of a fundamental system for the differential equation and particular solutions for polynomial right-hand sides. A different method to construct local approximation spaces is based on changes of variables. If the change of variables $x \mapsto \tilde{x}$ maps the problem onto a problem which can be approximated well (in some appropriate norm) by polynomials (in $\tilde{x}$), say, then the 'mapped polynomials', i.e. $P(\tilde{x}(x))$, where P is a polynomial, also have good approximation properties. For example, Babuška *et al.*[8] considered the problem

$$-\partial_x(a(x,y)\,\partial_x u) - \partial_y(a(x,y)\,\partial_y u) = f \quad \text{on } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

where the coefficient $a(x, y)$ is assumed to satisfy

$$0 < \alpha \leqslant a(x,y) \leqslant \beta < \infty$$

and is unidirectionally rough, i.e. the coefficient $a(x, y)$ is smooth in the y direction whereas it is rough in the x direction. The roughness of the coefficient $a(x, y)$ results in poor regularity properties of the solution u, and thus the usual finite element method leads to mesh sizes h which are prohibitively expensive. For the simplified model, $a(x, y) = a(x)$, the change of variables

$$\tilde{x} = \int_0^x \frac{\mathrm{d}t}{a(t)}$$
$$\tilde{y} = y$$

transforms the problem into one for which a better regularity theorem holds: if $f \in L^2(\Omega)$, then the transformed function $\tilde{u}$ is in $H^2(\tilde{\Omega})$ ($\tilde{\Omega}$ denotes the image of Ω under the above transformation; cf. Reference 8 for a proof). Thus, $\tilde{u}$ can be approximated by linear functions in $\tilde{x}$, $\tilde{y}$. Formulating in the original co-ordinates gives that u can be approximated on the patch Ω_j by

$$v_j \in \operatorname{span}\left\{1,\ \int_0^x \frac{\mathrm{d}t}{a(t)},\ y\right\}$$

such that

$$\| u - v_j \|_{H^1(\Omega_j)} \leqslant C(\operatorname{diam}\Omega_j)\,\| \tilde{u} \|_{H^2(\tilde{\Omega}_j)}$$

($\tilde{\Omega}_j$ is the image of the patch Ω_j under the transformation). The constant $C > 0$ depends only on α, β and is independent of the roughness of the coefficient $a(x)$, and thus these local spaces have good approximation properties on patches independent of the bad behaviour the coefficient $a(x)$ might display.

Let us finally point out that in this example the change of variables can be done locally: if $(x_j, y_j) \in \Omega_j$, then the approximating functions can be chosen to be in $\text{span}\{1, \int_{x_j}^{x} 1/a(t), y\}$.

Another instance where the idea of using a change of variables is successfully used can be found in References 6 and 7. For elliptic problems in two dimensions with corners or interfaces, the use of a conformal map is proposed which maps the rough solution to a smoother function. This smoother function on the mapped domain can be approximated by polynomials. Hence, the images of polynomials under the inverse of this conformal map are used for the approximation of the original problem.

5.2. Optimality of local approximation spaces and n-width

An interesting issue in the context of finding good local approximation spaces is the question of optimality of local spaces. We measure optimality in terms of n-width,[35] i.e. in terms of error per degree of freedom for a whole class of functions:

$$d(n, \|\cdot\|, S) = \inf_{E_n} \sup_{f \in S} \inf_{g \in E_n} \|f - g\|$$

where E_n denotes an n-dimensional space, and S is the class of functions that we wish to approximate; typically, S is chosen as the unit ball of some appropriate Banach space. A minimizing space E_n is called an optimal space. We see that this notion of optimality depends on the dimension n, the norm $\|\cdot\|$, in which we measure the approximation error, and the choice of the class S. In particular, different classes S lead to different optimal spaces. In practice, of course, we want robust optimal (or near-optimal) approximation spaces because we might not know with respect to which class of functions we should optimize (this uncertainty issue is elaborated in Reference 36). For example, if we choose $\|\cdot\| = \|\cdot\|_{H^1(\Omega)}$, and if we are interested in approximating functions which are analytic on $\tilde{\Omega} \supset\supset \Omega$, the class S could be taken as the unit ball of any $H^k(\Omega)$, $k > 1$. Thus, due to this uncertainty, we want the approximation spaces to be optimal for as large a class of functions as possible.

Proposition 2 below exhibits an example of approximation spaces which are optimal for large classes of harmonic functions. In the framework of the PUM, Proposition 2 yields the following result: For the approximation of harmonic functions on patches Ω_i which are discs $\subset \mathbb{R}^2$, the choice of spaces of harmonic polynomials as local approximation spaces V_i is an optimal one. The notion of optimality here is tied to the assumptions of Proposition 2, namely, the restriction to functions defined on discs and to rotationally invariant classes of harmonic functions.

For ease of exposition, we deal with complex-valued, holomorphic (analytic) functions and observe that the case of harmonic functions follows by taking real parts.
We introduce the spaces

$$\mathscr{H}^k = \{f \in H^k(B_R(0)) \,|\, f \text{ is holomorphic on} B_R(0)\}, \quad k \geqslant 0$$

and a Hilbert space $\mathscr{H}$ of holomorphic functions with inner product $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ and norm $\|\cdot\|_{\mathscr{H}}$ is called *rotationally invariant* if

$$\|f(az)\|_{\mathscr{H}} = \|f(z)\|_{\mathscr{H}} \quad \forall f \in \mathscr{H}, a \in \mathbb{C}, |a| = 1$$

The space $\mathscr{H}^0$ ($\mathscr{H}^1$) is a Hilbert space with the L^2 (H^1) inner product and thus a closed subspaces of $L^2(B_R(0))$ ($H^1(B_R(0))$). Therefore, the space $L^2(B_R(0))$ ($H^1(B_R(0))$) can be written as the direct sum of $\mathscr{H}^0$ ($\mathscr{H}^1$) and its orthogonal complement. This reduces the search for optimal spaces for

the approximation of holomorphic functions in the L^2 (H^1) norm to the problem of finding optimal subspaces of $\mathscr{H}^0$ ($\mathscr{H}^1$).

The polynomials $(z^n)_{n=0}^{\infty}$ form an orthogonal basis of $\mathscr{H}^k$, $k \geqslant 0$, and it is easy to see that they actually form an orthogonal basis for any rotationally invariant Hilbert space of holomorphic functions. Therefore, setting $\psi_H(n) = < z^n, z^n >_{\mathscr{H}}$, gives the representation

$$\| f \|_{\mathscr{H}}^2 = \sum_{n=0}^{\infty} |f_n|^2 \psi_{\mathscr{H}}(n)$$

where the f_n are the Taylor coefficients of the holomorphic function f, i.e.

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \quad \text{on } B_R(0)$$

For example, we have

$$\psi_{\mathscr{H}^0}(n) = \pi \frac{1}{n+1} R^{2n+2}$$

$$\psi_{\mathscr{H}^1}(n) = \pi R^{2n} \left(\frac{R^2}{n+1} + n \right)$$

Proposition 2. *Let $\mathscr{H}_1, \mathscr{H}_2$ be two rotationally invariant Hilbert spaces of holomorphic functions on $B_R(0)$. Assume that the quotient*

$$\frac{\psi_{\mathscr{H}_1}(n)}{\psi_{\mathscr{H}_2}(n)}$$

is monotonically decreasing in n. Then the spaces

$$T_p = \operatorname{span}\{z^n \mid n = 0, \ldots, p\}$$

are optimal spaces for the approximation of functions in $\mathscr{H}_2$ in the $\| \cdot \|_{\mathscr{H}_1}$ norm, i.e. the space T_p minimizes the expression

$$\sup_{f \in \mathscr{H}_2} \inf_{g \in E_p} \frac{\| f - g \|_{\mathscr{H}_1}}{\| f \|_{\mathscr{H}_2}} \tag{20}$$

over all p-dimensional subspaces E_p of $\mathscr{H}_1$.

Proof. The proof proceeds in two steps. First, we will see that (20) is bigger than or equal to

$$\left(\frac{\psi_{\mathscr{H}_1}(p+1)}{\psi_{\mathscr{H}_2}(p+1)} \right)^{1/2}$$

for any p-dimensional subspace of $\mathscr{H}_1$. In the second step, we see that this infimum is attained for the choice of T_p as p-dimensional approximation space. Let a p-dimensional subspace E_p of $\mathscr{H}_1$ be given. Choose $f \in T_{p+1}$ orthogonal (with respect to $\langle \cdot, \cdot \rangle_{\mathscr{H}_1}$) to E_p. Then, the square of (20) can be bounded from below by

$$\frac{\| f \|_{\mathscr{H}_1}^2}{\| f \|_{\mathscr{H}_2}^2} \geqslant \inf_{f \in T_{p+1}} \frac{\| f \|_{\mathscr{H}_1}^2}{\| f \|_{\mathscr{H}_2}^2} \geqslant \frac{\psi_{\mathscr{H}_1}(p+1)}{\psi_{\mathscr{H}_2}(p+1)}$$

where we made use of the monotonicity assumption. On the other hand, the choice $E_p = T_p$ implies

$$\sup_{f \in \mathscr{H}_2} \inf_{g \in T_p} \frac{\|f - g\|^2_{\mathscr{H}_1}}{\|f\|^2_{\mathscr{H}_2}} \leqslant \sup_{f \in \mathscr{H}_2} \frac{\sum_{n=p+1}^{\infty} |f_n|^2 \psi_{\mathscr{H}_1}(n)}{\sum_{n=0}^{\infty} |f_n|^2 \psi_{\mathscr{H}_2}(n)} \leqslant \frac{\psi_{\mathscr{H}_1}(p+1)}{\psi_{\mathscr{H}_2}(p+1)}$$

where we made again use of the monotonicity assumption. □

Choosing $\mathscr{H}_1$ in Proposition 2 to be $\mathscr{H}^0$ or $\mathscr{H}^1$ shows that the spaces T_p are optimal if we measure approximability in the L^2 or H^1 norm and if we approximate rotationally invariant classes of functions which satisfy a certain monotonicity of the numbers $\psi_{\mathscr{H}_2}(n)$. All spaces $\mathscr{H}^k$ fall into this latter category and many spaces of holomorphic functions which are in some weighted H^k spaces where the weight is rotationally symmetric. Let us further note that in the context of the PUM, Theorem 1 suggests that we optimize with respect to the norm $(\operatorname{diam}^2(\Omega_i)\, |\cdot|^2_{H^1(\Omega_i)} + \|\cdot\|^2_{L^2(\Omega_i)})^{1/2}$. The proof of Proposition 2 shows that this choice of norm also leads to the spaces T_p as optimal approximation spaces.

Remark 4: As stated earlier, Proposition 2 can be formulated for harmonic functions as well. Then, the $(2p+1)$-dimensional spaces of harmonic polynomials are optimal under similar conditions. For example, the $(2p+1)$-dimensional spaces of harmonic polynomials are optimal for the approximation of harmonic function on the discs $B_R(0)$ which are in the spaces $H^k(B_R(0))$, $k \geqslant 1$.

Remark 5: Proposition 2 and the preceding remark state (loosely speaking) that harmonic polynomials are universally optimal for the approximation of harmonic functions on discs. This is partly a justification for the approximation with harmonic polynomials in Section 7.1: As long as the patches differ not too much from discs, we expect spaces of harmonic polynomials to be nearly optimal for the approximation of harmonic functions.

Let us stress here that harmonic polynomials are no longer optimal if one of the assumptions of Proposition 2 is changed. For example, consider approximation on a sector W with angle ω and size R (for notational convenience, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$):

$$W = \{z \in \mathbb{C} \,|\, |z| < R \text{ and } 0 < \arg z < \omega\}.$$

Assume that we are interested in approximating (in H^1, say) harmonic functions satisfying homogeneous Dirichlet conditions on the two straight sides of the sector, i.e. functions of the form

$$u = \sum_{n=1}^{\infty} a_n \operatorname{Im} z^{\pi n/\omega}$$

with coefficients $a_n \in \mathbb{R}$. Then the functions $\operatorname{Im} z^{\pi n/\omega}$, $n = 1, \ldots, p$ form optimal spaces of dimension p for the whole scale of spaces

$$\tilde{\mathscr{H}}^k = \left\{ u = \sum_{n=1}^{\infty} a_n \operatorname{Im} z^{\pi n/\omega} \,\middle|\, a_n \in \mathbb{R} \quad \text{and} \quad \sum_{n=1}^{\infty} |a_n|^2 (1+n)^{2k-1} R^{2n\pi/\omega} < \infty \right\}, \quad k > 1.$$

The proof of this statement is very similar to the proof of Proposition 2. A different way of defining the spaces $\tilde{\mathscr{H}}^k$ is to say that harmonic functions in $H^k(B_{R^{\pi/\omega}}(0))$ which are antisymmetric with respect to the real axis are mapped onto the elements of $\tilde{\mathscr{H}}^k$ under the conformal change of variables $z \mapsto z^{\omega/\pi}$.

6. THE PUM IN TWO DIMENSIONS

In the two-dimensional case—just as in the one-dimensional one—we have to address the creation of a partition of unity and the choice of local approximation spaces. Let us first outline two different types of partitions of unity. If a domain $\tilde{\Omega} \supseteq \Omega$ is given by a mesh (i.e., triangles, quadrilaterals, or mapped triangles or quadrilaterals), then the usual pyramid functions associated with the nodes of the mesh form a piecewise smooth partition of unity. Since in all the numerical examples below, we use this kind of partition of unity, let us exemplify this idea with one example. Let Ω be the unit square $(0, 1) \times (0, 1)$ and let it be subdivided into n^2, $n \in \mathbb{N}$, squares of side length $h = 1/n$ with nodes (x_j, y_j), $j = 1, \ldots, (n+1)^2$. Define

$$\varphi(x) = \begin{cases} (1-x)(1-y) & \text{for } (x, y) \in [0,1] \times [0,1] \\ (1+x)(1-y) & \text{for } (x, y) \in [-1, 0] \times [0,1] \\ (1+x)(1+y) & \text{for } (x, y) \in [-1, 0] \times [-1, 0] \\ (1-x)(1+y) & \text{for } (x, y) \in [0, 1] \times [-1, 0] \\ 0 & \text{elsewhere} \end{cases} \tag{21}$$

Then the functions $\varphi_j(x) = \varphi((x - x_j)/h, (y - y_j)/h)$ associated with the $(n+1)^2$ patches $\Omega_j = \{(x, y) \,|\, |x - x_j| < h, |y - y_j| < h\}$ form a partition of unity. This is the analogous construction to the first construction of Section 4.2.

The second type of partition of unity is given by the construction described in the fifth method of Section 4.2. For example, if Ω is covered by circles, ellipses, or quadrilaterals, it is easy to construct a partition of unity of any desired regularity by the 'normalizing' technique outlined in the fifth method of Section 4.2. Let us stress at this point that the partition of unity does not have to be related to the geometry of the domain of interest.

Many of the observations of Section 4.3 about the one-dimensional case are true in the two-dimensional setting as well. For example, it can be shown that the piecewise bilinear partition of unity described above in conjunction with polynomial local approximation spaces V_j displays the same difficulties with linear dependencies as the space $V^{\mathrm{II},1}$ of Section 4.3 (cf. Reference 34). However, the same idea of modifying the partition of unity on patches close to the boundary as is proposed in the third method of Section 4.2 leads to a basis of the finite element space which is directly related to the bases of the local spaces. As observed in the one dimensional case, the stiffness matrix resulting from the modified partition of unity can actually be constructed from the original one.

Related to the choice of the partition of unity (and the local approximation spaces) is the question of integrating the shape functions against each other, because the partition of unity is typically only piecewise smooth (and hence the shape functions). This issue will be explored in more details in a forthcoming paper. For all the numerical examples below, we use the partition of unity for the unit square described above, and therefore the usual integration schemes on each of the n^2 square can be applied.

Another important question is the implementation of essential boundary conditions. For some problems, it is easy to construct local approximation spaces V_j on patches close to the boundary which have both good approximation properties and satisfy the essential boundary conditions. This is the case in the one-dimensional problem (10) with the choice V_0^{I}. For an example in two dimension, consider the implementation of homogeneous Dirichlet conditions on a straight part of the boundary for the problem $-\Delta u = 0$. Here, harmonic polynomials which are antisymmetric

with respect to that straight line have good approximation properties and satisfy the homogeneous boundary conditions. A similar approach works in a corner.

One way to imitate the way essential boundary conditions are implemented in the classical finite element methods is to use spaces of (piecewise) full polynomials on patches close to the boundary. In that case, all the techniques of the usual finite element methods can be applied. Another approach to the implementation of essential boundary conditions is the use of Lagrange multipliers or a penalty method. In the numerical examples below, we chose the boundary conditions to be natural in order to be able to concentrate on the approximation properties of the spaces constructed with the PUM.

7. NUMERICAL EXAMPLES

In this section, we will present two numerical examples, namely, the approximation of solutions to Laplace's equation and Helmholtz's equation on the unit square with the PUM.

7.1. Laplace's equation

Let us consider first approximations to the solution of

$$-\Delta u = 0 \quad \text{on } \Omega = (0,1)\times(0,1)$$

$$\partial_n u = \partial_n \operatorname{Re}\left(\frac{1}{a^2 - z^2} + \frac{1}{a^2 + z^2}\right) \quad \text{on } \partial\Omega, \qquad a = 1{\cdot}05$$

and we fix u in $(0,0)$ in order to make the solution of this problem unique. Since we want to present a p version of the PUM where the local approximation spaces are chosen as spaces of harmonic polynomials of degree p, we need to clarify the approximation properties of harmonic polynomials. This is done in the following two theorems. Note that there are only $2p+1$ harmonic polynomials of degree p.

Theorem 2 (Szegö). *Let $\Omega \subset \mathbb{R}^2$ be a simply connected, bounded Lipschitz domain. Let $\tilde{\Omega} \supset\supset \Omega$ and assume that $u \in L^2(\tilde{\Omega})$ is harmonic on $\tilde{\Omega}$. Then there is a sequence $(u_p)_{p=0}^{\infty}$ of harmonic polynomials of degree p such that*

$$\|u - u_p\|_{L^\infty(\Omega)} \leqslant C\mathrm{e}^{-\gamma p}\|u\|_{L^2(\tilde{\Omega})}$$

$$\|\nabla(u - u_p)\|_{L^\infty(\Omega)} \leqslant C\mathrm{e}^{-\gamma p}\|u\|_{L^2(\tilde{\Omega})}$$

where γ, $C > 0$ depend only on Ω, $\tilde{\Omega}$.

Proof: See References 45 and 37. □

Theorem 3. *Let Ω be a simply connected bounded Lipschitz domain, star-shaped with respect to a ball. Let the exterior angle of Ω be bounded from below by $\lambda\pi$, $0 < \lambda < 2$. Assume that $u \in H^k(\Omega)$, $k > 1$, is harmonic. Then there is a sequence $(u_p)_{p=2}^{\infty}$ of harmonic polynomials of degree p such that*

$$\|u - u_p\|_{H^j(\Omega)} \leqslant C\left(\frac{\ln p}{p}\right)^{\lambda(k-j)}\|u\|_{H^k(\Omega)}, \quad j = 0, \ldots, [k]$$

where $C > 0$ depends only on Ω and k.

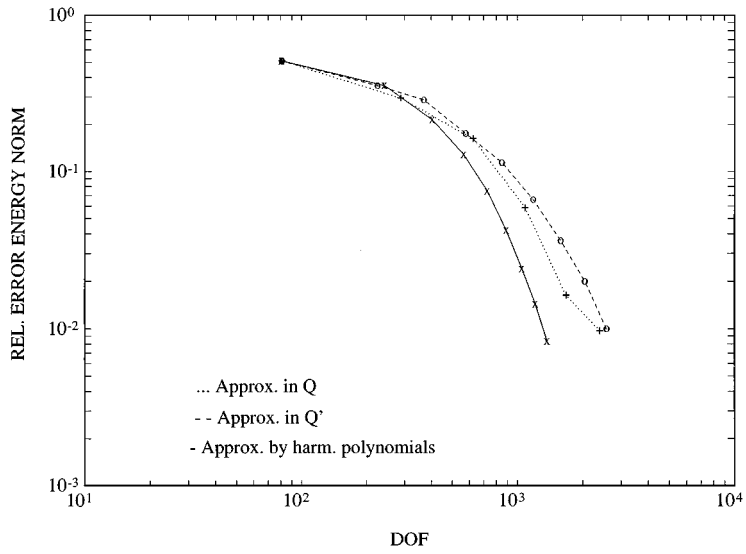

Figure 1. PUM, classical p version for Laplace's equation; $a = 1{\cdot}05$, 8×8 elements

See Reference 5 for a proof of Theorem 3. Note that typically $\lambda \leqslant 1$ and that for domains with re-entrant corners, λ can be significantly less than 1.

Remark 6: The restriction in Theorem 3 that Ω be star-shaped with respect to a ball is not a big constraint for our purposes because we are interested in local estimates on patches and the patches are typically chosen to be star-shaped.

For the PUM the domain Ω is covered by square patches and the partition of unity is chosen to be piecewise bilinear as described in Section 6. The specific choice $n = 8$ is made, and the local approximation spaces V_j consist of harmonic polynomials of degree p (p ranging from 0 to 8). In Figure 1 we plot the relative error in energy norm (i.e. the relative error in the H^1 semi-norm) versus the number of unknowns for three methods. The PUM is compared with two classical p versions, namely, the tensor product spaces Q_p and the serendipity spaces Q'_p based on an 8×8 mesh. We see clearly that the use of harmonic polynomials made possible by the PUM is advantageous: in order to achieve 1 per cent error in H^1, the PUM based on harmonic polynomials needs only half as many DOF as the usual p, version spaces Q_p, Q'_p. This is in accordance with our earlier observation that the number of harmonic polynomials grows linearly with the degree p, whereas the size of full polynomial spaces grows quadratically. Note that the disparity between the PUM and the spaces of full polynomials becomes bigger for higher accuracy. See Reference 34 for a more detailed study of the performance of the PUM as the parameters n and a are varied.

Remark 7. For the elasticity equations in two dimensions, the situation is completely analogous to Laplace's equation. In the absence of body forces, the displacement field (u, v)

under the plane strain assumption can be expressed by two holomorphic functions φ, ψ (see Reference 38):

$$2\mu(u + \mathrm{i}v) = \kappa\varphi(z) - z\overline{\varphi'(z)} - \bar{\psi}(z) \tag{22}$$

where $\kappa = (\lambda + 3\mu)/(\lambda + \mu)$ and λ, μ are the Lamé constants. Choosing $\kappa = (\lambda^* + 3\mu)/(\lambda^* + \mu)$ with $\lambda^* = 2\lambda\mu/(\lambda + 2\mu)$ gives the representation for the case of plane stress. The holomorphic functions φ, ψ can be approximated by complex polynomials φ_p, ψ_p of degree p, and thus the functions

$$\kappa\varphi_p(z) - z\bar{\varphi}_p(z) - \bar{\psi}_p(z)$$

take the rôle of 'harmonic' polynomials for the elasticity equations. It can be shown that Theorems 2 and 3 hold verbatim for the approximation of the solutions to the elasticity equations with these 'generalized harmonic polynomials'.[5]

7.2. Helmholtz's equation

The next numerical example deals with the approximations to Helmholtz's equation. On the unit square, we consider

$$\begin{aligned} -\Delta u - k^2 u &= 0 \quad \text{on } \Omega = (0, 1)\times(0, 1) \\ \partial_n u + \mathrm{i}ku &= g \quad \text{on } \partial\Omega \end{aligned} \tag{23}$$

where g is chosen such that the exact solution is a plane wave of the form

$$u(x, y) = \exp\{\mathrm{i}k(x\cos\theta + y\sin\theta)\}, \quad \theta = \frac{\pi}{16}$$

The following two types of local approximation spaces were analysed in Reference 5. The first type are 'generalized harmonic polynomials' as alluded to in the introduction. Written in polar co-ordinates, they take the form

$$V^V(p) = \operatorname{span}\{\mathrm{e}^{\pm \mathrm{i}n\theta} J_n(kr) | n = 0, \ldots, p\} \tag{24}$$

where the functions J_n are the Bessel functions of the first kind (see e.g. Reference 39). The second type are systems of plane waves given by

$$W(p) = \operatorname{span}\left\{\exp(\mathrm{i}k(x\cos\theta_j + y\sin\theta_j)) \,|\, \theta_j = \frac{2\pi}{p} j, j = 0, \ldots, p-1\right\} \tag{25}$$

Remark 8: The spaces $V^{\mathrm{V}}(p)$, the spaces of 'generalized harmonic polynomials', share the optimality properties of the harmonic polynomials for the approximation of harmonic functions on discs (see Section 5.2); the spaces $V^{\mathrm{V}}(p)$ are optimal in the sense of n-width for large classes of rotationally invariant spaces of solutions of Helmholtz's equation on discs.

Remark 9: The numerical examples below are based on the spaces $W(p)$. In all computations we chose p to be of the form $2 + 4m$, $m \in \mathbb{N}_0$, so that the exact solution of problem (23) is not an element of the PUM space.

The approximation properties of these two types of spaces are very similar to the usual harmonic polynomials. In fact, we have

Theorem 4. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected, bounded Lipschitz domain. Let $\tilde{\Omega} \supset\supset \Omega$ and assume that $u \in L^2(\tilde{\Omega})$ solves the homogeneous Helmholtz equation on $\tilde{\Omega}$. Then*

$$\inf_{u_p \in V^{\mathrm{V}}(p)} \|u - u_p\|_{H^1(\Omega)} \leqslant C\mathrm{e}^{-\gamma p}\|u\|_{L^2(\tilde{\Omega})}$$

$$\inf_{w_p \in W(p)} \|u - w_p\|_{H^1(\Omega)} \leqslant \tilde{C}e^{-\gamma p/\ln p}\|u\|_{L^2(\tilde{\Omega})}$$

where C, $\tilde{C}$, γ, and $\tilde{\gamma}$ depend only on Ω, $\tilde{\Omega}$, and k.

Remark 10: For the solution of the model problem (23), Theorem 4 can be strengthened:

$$\inf_{u_p \in V^{\mathrm{V}}(p)} \|u - u_p\|_{H^1(\Omega)} \leqslant C(\gamma, \Omega, k)\mathrm{e}^{-\gamma p}$$

$$\inf_{u_p \in W(p)} \|u - w_p\|_{H^1(\Omega)} \leqslant C(\gamma, \Omega, k)\mathrm{e}^{-\gamma p}$$

holds for any fixed $\gamma > 0$.

Theorem 5. *Let Ω be a simply connected bounded Lipschitz domain, star-shaped with respect to a ball. Let the exterior angle of Ω be bounded from below by $\lambda\pi$, $0 < \lambda < 2$. Assume that $u \in H^s(\Omega)$, $s > 1$, satisfies the homogeneous Helmholtz equation. Then*

$$\inf_{u_p \in V^{\mathrm{V}}(p)} \|u - u_p\|_{H^j(\Omega)} \leqslant C_j \left(\frac{\ln p}{p}\right)^{\lambda(s-j)} \|u\|_{H^k(\Omega)}, \quad j = 0, \ldots, [s]$$

$$\inf_{u_p \in W(p)} \|u - u_p\|_{H^j(\Omega)} \leqslant C_j \left(\frac{\ln^2 p}{p}\right)^{\lambda(s-j)} \|u\|_{H^k(\Omega)}, \quad j = 0, \ldots, [s]$$

The PUM can be based on either approximation space. In the numerical results below, we concentrate on the PUM based on the spaces $W(p)$ of plane waves (for a comparison with the 'generalized harmonic polynomials' $V^{\mathrm{V}}(p)$, see Reference 5). The domain Ω is covered by square patches and the partition of unity consists again of piecewise bilinear functions as described in Section 6. The local approximation spaces V_j are taken as the spaces $W(p)$.

Remark 11: The theorems above merely address the issue of approximability; we do not deal with the delicate question of stability of the finite element methods based on these spaces. Suffice it to say that the spaces created by the PUM are stable under the assumption that the mesh size h is sufficiently small with respect to the wave number k (see Reference 5). However, as can be seen in the numerical results, the PUM performs very well as a p version for very coarse meshes.

In Tables I–VI the PUM is compared with the usual Galerkin finite element method (FEM), the generalized least-squares finite element method (GLSFEM) of Reference 40, and the quasi-stabilized finite element method (QSFEM) of Reference 41. Since all three methods are based on piecewise linear functions on uniform grids, Tables I and II include the piecewise linear best approximant for reference. The FEM, GLSFEM, and QSFEM differ in their choice of the bilinear form. In particular, the bilinear form of the QSFEM is constructed such that 'pollution' (see Reference 41) is minimized, and thus the QSFEM is virtually the best method available which is based on piecewise linear functions. We will see that the PUM compares very favourably with the QSFEM.

Table I. DOF necessary to obtain accuracy ε in L^2 norm; $k = 100$

ε (%)	best approximant	QSFEM	GLSFEM	FEM
30	2·045D + 3	3·969D + 3	2·016D + 4	7·784D + 4
10	5·041D + 3	1·000D + 4	6·150D + 4	2·352D + 5
5	8·464D + 3	1·960D + 4	1·274D + 5	4·692D + 5

Table II. DOF necessary to achieve various accuracies in L^2 for PUM with $n = 4$ and various other methods; $k = 100$

p	L^2 error (%)	PUM	Best approx.	QSFEM	FEM
26	10·8	6·50D + 2	3·80D + 3	7·95D + 3	2·08D + 5
30	0·69	7·50D + 2	5·89D + 4	1·23D + 5	3·23D + 6
34	0·11	8·50D + 2	3·45D + 5	7·23D + 5	1·90D + 7

Table III. Number of operations using band elimination — the p version of the PUM; $n = 4$; $k = 100$; error in L^2

p	L^2 error (%)	PUM	QSFEM	FEM
26	10·8	1·76D + 7	6·3D + 7	4·3D + 11
30	0·69	2·71D + 7	1·5D + 10	1·01D + 13
34	0·11	3·94D + 7	5·2D + 11	3·6D + 14

We discuss the cases $k = 100$ with the L^2 norm as the error measure and $k = 32$ with the H^1 as the error measure. Tables I–IV show the performance of the PUM in comparison with the other methods for $k = 100$ and the L^2 norm as error measure. Table I shows the number of DOF needed to achieve a certain L^2 accuracy for the various piecewise linear methods. We see that the QSFEM needs 2 times as many DOF as the best approximant, while the GLSFEM needs 10–15 and the FEM 40–50 as many. Table II shows that the p version of the PUM can achieve the same accuracy as the other methods with markedly fewer DOF. This can be attributed to the exponential approximability of the PUM: According to Remark 10 the approximation properties of the PUM space based on plane waves $W(p)$ are exponential in p, whereas the h versions of the piecewise linear methods can only have algebraic rates of convergence. This explains why the discrepancy between the PUM and the other methods becomes more pronounced for high accuracy: in order to achieve 10 per cent accuracy in L^2, the best approximant needs 6 times as many DOF as the PUM, whereas it needs 400 times as many as the PUM to achieve 0·11 per cent accuracy. Table III shows how this reduction of DOF translates into a reduction in the operation count if a direct solver (band elimination) is used. Again, the PUM outperforms the QSFEM and the FEM for the case of 10·8 per cent accuracy and is greatly superior for high accuracy.

In Tables I–III we saw the performance of the PUM as a p version. Table IV shows the performance of the PUM as an hp version by listing the number of operations for the band

Table IV. Number of operations for *hp* version of PUM; $k = 100$; L^2 error

p	n	L^2 error (%)	NOP PUM
26	4	10·8	1·76D + 7
18	8	10·6	5·23D + 7
14	16	9·5	2·75D + 8

Table V. Operation count for solving linear system; error in H^1 norm; $k = 32$

	Galerkin			QSFEM	
$\sqrt{\text{DOF}}$	H^1 error (%)	No. iter	NOP	H^1 error (%)	No. iter	NOP
32	65	232	4·51D + 6	30·5	272	5·29D + 6
64	21·7	434	3·37D + 7	14·3	492	3·82D + 7
128	8·16	831	2·68D + 8	7·02	953	2·96D + 8
256	3·64	1665	2·07D + 9	3·48	1863	2·31D + 9
512	1·72	3263	1·62D + 10	1·69	3752	1·86D + 10

Table VI. Operation count for band elimination for PUM; $k = 32$, error in H^1; $n = 1$

p	H^1 error (%)	NOP PUM
18	46	1·3D + 5
22	6·7	2·3D + 5
26	0·38	3·8D + 5
30	0·00025	5·9D + 5

elimination for various combinations of p and $h = 1/n$ which result in an accuracy of ca. 10 per cent in L^2. We see that the operation count increases with n (and thus with decreasing p). This can again be explained by the fact that the PUM spaces feature exponential approximability as p versions but only algebraic approximability as h versions.

Tables V and VI illustrate the case $k = 32$ with the H^1 semi-norm as the error measure. The linear system of the usual FEM and the QSFEM is solved using the iterative scheme proposed by Freund[42]. We compare the cost of these iterative schemes (Table V) with the cost of the band elimination for the PUM (Table VI) as a p version ($n = 1$). We see that the PUM is cheaper than the QSFEM, which is virtually the optimal method for piecewise linear ansatz functions. The PUM is cheaper in the whole range of accuracies (50–0 per cent). As in the case of DOF versus L^2 accuracy above, the disparity between the PUM and the other methods becomes bigger for high accuracy: for 50 per cent error, the PUM is 30 times cheaper than the FEM, and for 1 per cent the PUM is 10^5 times cheaper!

Remark 12: In the operation count for the PUM (Tables IV and VI) only the contributions of the band elimination are reported. This is justified by the particular structure of the problem under consideration. The mesh is uniform, the partition of unity consists of piecewise bilinear

functions and the local approximation spaces are spaces of plane waves. All of this can be exploited in the construction of the stiffness matrix, and the resulting cost of the generation the stiffness matrix is of lower order compared with the cost of the linear solver.

The numerical examples show that the PUM performs much better than the usual h versions both in terms of error versus DOF and error versus operation count. This is due to the fact that the PUM allows us to use local approximation spaces that capture the local behaviour of the solution very well, even if the solution is rough. In this example, the approximation with plane waves is very efficient although the wave number k is large ($k = 32$, $k = 100$). We saw that the PUM outperforms the h version for accuracies of practical interest (50–1 per cent in H^1, say) and that the PUM is immensely superior for high accuracy.

8. *A POSTERIORI* ERROR ESTIMATION

A posteriori error estimation for finite element solutions obtained by the PUM is possible if local problems on the patches $\Omega_i \cap \Omega$ can be solved (or suitably approximated). In order to demonstrate this, let us consider the model problem

$$\begin{aligned} Lu = -\operatorname{div} a(x) \operatorname{grad} u + c(x)\, u = f \in L^2(\Omega) \qquad & \text{on } \Omega \\ u = 0 \qquad & \text{on } \Gamma_D \subset \partial\Omega,\ \Gamma_D \neq \emptyset \\ \sigma_n u = a(x)\partial_n u = g \in H^{-1/2}(\Gamma_N) \quad & \text{on } \Gamma_N = \partial\Omega \backslash \Gamma_D \end{aligned} \tag{26}$$

where a, c are bounded functions and satisfy the inequality

$$0 < \alpha \leqslant \min(a(x), c(x)) \leqslant \max(a(x), c(x)) \leqslant \beta < \infty$$

The weak form of this problem is to find $u \in H^1_D(\Omega)$ such that

$$B(u, v) = F(v) \quad \forall v \in H^1_D(\Omega) = \{v \in H^1(\Omega) \,|\, v = 0 \text{ on } \Gamma_D\} \tag{27}$$

where the bilinear form B and the linear functional F are defined in the standard way. The conditions on the coefficients a, c imply that

$$\alpha \| u \|^2_{H^1(\Omega)} \leqslant B(u, u)$$

$$|B(u, v)| \leqslant \beta \| u \|_{H^1(\Omega)} \| v \|_{H^1(\Omega)}$$

Let V_{FE} be a conforming PUM space, i.e. $V_{\mathrm{FE}} \subset H^1_D(\Omega)$. Then, the finite element solution $u_{\mathrm{FE}} \in V_{\mathrm{FE}}$ is defined by

$$B(u_{\mathrm{FE}}, v) = F(v) \quad \forall v \in V_{\mathrm{FE}} \subset H^1_D(\Omega) \tag{28}$$

On each patch $\Omega_i \cap \Omega$, we introduce the local problem

$$\text{find } \eta_i \in W_i \quad B(\eta_i, v) = B(u - u_{\mathrm{FE}}, v) \quad \forall v \in W_i \tag{29}$$

where

$$W_i = \{v \in H^1(\Omega_i \cap \Omega) \,|\, v = 0 \text{ on } \partial(\Omega_i \cap \Omega) \backslash \Gamma_N\} \tag{30}$$

Remark 13: If we require the PUM space to be of degree 2, i.e. if $V_{\mathrm{FE}} \subset H^1_D(\Omega) \cap C^2(\Omega)$, integration by parts allows us to express the right-hand side of (29) explicitly in terms of the given data L, f, and g:

$$B(\eta_i, v) = B(u - u_{\mathrm{FE}}, v) = \int_{\Omega_i \cap \Omega} (f - Lu_{\mathrm{FE}})\, v \, \mathrm{d}x + \int_{\Gamma_N} (g - \sigma_n u_{\mathrm{FE}})\, v \, \mathrm{d}s \tag{31}$$

In this last integration by parts argument, we made use of the assumption $V_{\mathrm{FE}} \subset C^2(\Omega)$. This is an important simplification in practice because in that way, the evaluation of the right-hand side of (29) requires only knowledge about u_{FE} and its derivatives on the patch $\Omega_i \cap \Omega$. If the space V_{FE} is less regular (e.g. $V_{\mathrm{FE}} \subset C(\Omega)$ and piecewise C^2) the integration by parts argument introduces additional terms related to the jumps of derivatives; restricting ourselves to the case $V_{\mathrm{FE}} \subset C^2(\Omega)$ removes the necessity to determine the points where these jumps may occur.

Before we proceed to prove Theorem 6, which relates the error of the finite element solution to the local functions η_i, we need to impose some approximation properties on the local approximation spaces V_i.

Definition 3. A collection V_i of local approximation spaces has the *uniform Poincaré property* if there is $C_p > 0$ independent of i such that

1. for i such that $\Omega_i \cap \Gamma_D = \emptyset$, V_i contains the constant functions and
   $\inf_{\lambda \in \mathbb{R}} \| v - \lambda \|_{L^2(\Omega_i \cap \Omega)} \leqslant C_P \operatorname{diam}(\Omega_i) \| v \|_{H^1(\Omega_i \cap \Omega)} \quad \forall v \in H^1(\Omega_i \cap \Omega)$
2. for i such that $\Omega_i \cap \Gamma_D \neq \emptyset$
   $\| v \|_{L^2(\Omega_i \cap \Omega)} \leqslant C_P \operatorname{diam}(\Omega_i) \| v \|_{H^1(\Omega_i \cap \Omega)} \quad \forall v \in \{ v \in H^1(\Omega_i \cap \Omega) \,|\, v = 0 \text{ on } \Gamma_D \}$

Theorem 6. *Let $\{\Omega_i\}$ be a cover of Ω and $\{\varphi_i\}$ a (M, C_∞, C_G) partition of unity subordinate to the cover $\{\Omega_i\}$. Let the local approximation spaces $\{V_i\}$ have the uniform Poincaré property and assume that $v_i = 0$ on Γ_D for $v_i \in V_i$ with $\Omega_i \cap \Gamma_D \neq \emptyset$. Then there is $C = C(\alpha, \beta, M, C_\infty, C_G, C_P) > 0$ (which is explicitly available from the proof below) such that*

$$C^{-1} \left(\sum_i \| \eta_i \|^2_{H^1(\Omega_i \cap \Omega)} \right)^{1/2} \leqslant \| u - u_{\mathrm{FE}} \|_{H^1(\Omega)} \leqslant C \left(\sum_i \| \eta_i \|^2_{H^1(\Omega_i \cap \Omega)} \right)^{1/2} \tag{32}$$

where u_{FE} and η_i are defined in equations (28) *and* (29).

Proof: The proof follows very closely Reference 43. First, we observe that the finite element space V_{FE} constructed by the PUM is conforming, i.e. $V_{\mathrm{FE}} \subset H^1_D(\Omega)$. Furthermore, we have $W_i \subset H^1_D(\Omega)$ by continuing the elements of W_i by zero on $\Omega \backslash (\Omega_i \cap \Omega)$.

By the coercivity of B, the orthogonality relation satisfied by u_{FE}, and the fact that $\sum_i \varphi_i \equiv 1$ on Ω, we have

$$\begin{aligned}
\alpha \| u - u_{\mathrm{FE}} \|^2_{H^1(\Omega)} &\leqslant B(u - u_{\mathrm{FE}}, u - u_{\mathrm{FE}}) \\
&= B(u - u_{\mathrm{FE}}, u - u_{\mathrm{FE}} - v_{\mathrm{FE}}) \quad \forall v_{\mathrm{FE}} \in V_{\mathrm{FE}} \\
&= B\left(u - u_{\mathrm{FE}}, \sum_i \varphi_i (u - u_{\mathrm{FE}} - v_i) \right) \qquad v_{\mathrm{FE}} = \sum_i \varphi_i v_i, \quad v_i \in V_i \\
&= \sum_i B(\eta_i, \varphi_i (u - u_{\mathrm{FE}} - v_i)) \\
&\leqslant \beta \left(\sum_i \| \eta_i \|^2_{H^1(\Omega_i \cap \Omega)} \right)^{1/2} \left(\sum_i \| \varphi_i (u - u_{\mathrm{FE}} - v_i) \|^2_{H^1(\Omega_i \cap \Omega)} \right)^{1/2}
\end{aligned}$$

where we made use of the fact that $\varphi_i (u - u_{\mathrm{FE}} - v_i) \in W_i \subset H^1_D(\Omega)$. The uniform Poincaré property gives the existence of $v_i \in \mathbb{R}$ such that

$$\| u - u_{\mathrm{FE}} - v_i \|_{L^2(\Omega_i \cap \Omega)} \leqslant \min(1, C_p \operatorname{diam}(\Omega_i)) \| u - u_{\mathrm{FE}} \|_{H^1(\Omega_i \cap \Omega)},$$

and thus we can estimate

$$\sum_i \| \varphi_i(u - u_{\mathrm{FE}} - v_i) \|^2_{H^1(\Omega_i \cap \Omega)} \leqslant \sum_i C_\infty^2 \| u - u_{\mathrm{FE}} - v_i \|^2_{L^2(\Omega_i \cap \Omega)}$$

$$+ 2C_\infty^2 \| \nabla(u - u_{\mathrm{FE}} - v_i) \|^2_{L^2(\Omega_i \cap \Omega)}$$

$$+ 2\frac{C_G^2}{\mathrm{diam}(\Omega_i)^2} \| u - u_{\mathrm{FE}} - v_i \|^2_{L^2(\Omega_i \cap \Omega)}$$

$$\leqslant \sum_i (3C_\infty^2 + 2C_G^2 C_p^2) \| u - u_{\mathrm{FE}} \|^2_{H^1(\Omega_i \cap \Omega)}$$

$$\leqslant M(3C_\infty^2 + 2C_G^2 C_p^2) \| u - u_{\mathrm{FE}} \|^2_{H^1(\Omega)}$$

where we used Lemma 2 below. This gives the upper estimate of (32). For the lower estimate, we use the fact that each $\eta_i \in W_i \subset H_D^1(\Omega)$ and thus

$$\sum_i \| \eta_i \|^2_{H^1(\Omega_i \cap \Omega)} \leqslant \alpha^{-1} \sum_i B(\eta_i, \eta_i)$$

$$\leqslant \alpha^{-1} \sum_i B(u - u_{\mathrm{FE}}, \eta_i) = \alpha^{-1} B\left(u - u_{\mathrm{FE}}, \sum_i \eta_i\right)$$

$$\leqslant \beta\alpha^{-1} \| u - u_{\mathrm{FE}} \|_{H^1(\Omega)} \left\| \sum_i \eta_i \right\|_{H^1(\Omega)}$$

$$\leqslant \beta\alpha^{-1} \| u - u_{\mathrm{FE}} \|_{H^1(\Omega)} \sqrt{M} \left(\sum_i \| \eta_i \|^2_{H^1(\Omega_i \cap \Omega)} \right)^{1/2}$$

where we made again use of Lemma 2 below. This concludes the proof of Theorem 6. □

Lemma 2. *Let Ω be an open set, $\{\Omega_i\}$ be an open cover of Ω satisfying the pointwise overlap condition*

$$\mathrm{card}\{i \mid x \in \Omega_i\} \leqslant M \quad \forall x \in \Omega$$

Let u, $u_i \in H^1(\Omega)$ be such that

$$\mathrm{supp}\ u_i \subset \mathrm{closure}(\Omega_i \cap \Omega)$$

Then

$$\sum_i \| u \|^2_{H^k(\Omega_i)} \leqslant M \| u \|^2_{H^k(\Omega),} \quad k = 0,1$$

$$\| \sum_i u_i \|^2_{H^k(\Omega)} \leqslant M \sum_i \| u_i \|^2_{H^k(\Omega \cap \Omega_i),} \quad k = 0, 1$$

Proof: We will show the case $k = 0$, the case $k = 1$ being handled similarly. Let χ_i be the characteristic function of the domain $\Omega_i \cap \Omega$. Then

$$\sum_i \int_{\Omega_i \cap \Omega} |u|^2 = \sum_i \int_\Omega \chi_i |u|^2 = \int_\Omega \sum_i \chi_i |u|^2 \leqslant M \int_\Omega |u|^2$$

This proves the first estimate. For the second estimate, we use the overlap condition and the condition on the supports of the functions u_i to see that for each $x \in \Omega$, the sum on the left-hand

side extends over not more than M terms. Therefore,

$$\int_\Omega \left(\sum_i u_i\right)^2 \leqslant \int_\Omega M \sum_i |u_i|^2 \leqslant M \sum_i \|u\|^2_{L^2(\Omega_i \cap \Omega)} \qquad \square$$

Remark 14: The proof of Theorem 6 shows that the uniform Poincaré property could be weakened. It is enough that the L^2 projections Π_i: $H^1(\Omega_i \cap \Omega) \to V_i$ satisfy

$$\|\Pi_i u\|_{H^1(\Omega_i \cap \Omega)} \leqslant C_P \|u\|_{H^1(\Omega i \cap \Omega)}$$

$$\|u - \Pi_i u\|_{L^2(\Omega_i \cap \Omega)} \leqslant C_P \operatorname{diam}(\Omega_i) \|u\|_{H^1(\Omega i \cap \Omega)}$$

Remark 15: The existence of the uniform Poincaré constant is related to a certain uniformity of shapes of the patches. More precisely, for any bounded domain D, the constant λ, defined by

$$\lambda^{-1/2} = \sup_{u \in H^1(D)} \inf_{\mu \in \mathbb{R}} \frac{\|u - \mu\|_{L^2(D)}}{\|\nabla u\|_{L^2(D)}}$$

is the second (i.e. the first non-zero) eigenvalue of the Neumann problem

$$-\Delta u = \lambda u \quad \text{on } D$$

$$\partial_n u = 0 \quad \text{on } \partial D$$

Remark 16: Let us note that a simple scaling argument shows that the uniform Poincaré constant of Definition 3 depends only on the shape of the patches and not the diameters. Thus, one simple way to enforce a uniform Poincaré property is to restrict the number of possible shapes of the patches $\Omega_i \cap \Omega$.

Let us outline sufficient conditions on the patches $\Omega_i \cap \Omega$ that guarantee the uniform Poincaré property of the local approximation spaces V_i based on the following lemma.

Lemma 3. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain, $u \in H^1(\Omega)$. Then*

$$\|u - \underline{u}\|_{L^2(\Omega)} \leqslant \left(\frac{\omega_n}{|\Omega|}\right)^{1-1/n} (\operatorname{diam}(\Omega))^n \|\nabla u\|_{L^2(\Omega)} \tag{33}$$

where $\underline{u}$ is the average of u over Ω

$$\underline{u} = \frac{1}{|\Omega|} \int_\Omega u \tag{34}$$

$|\Omega|$ stands for the volume of Ω, and ω_n is the surface of the unit sphere in $\mathbb{R}^n$.

Proof: From Section 7.8 of Reference 44. $\square$

For patches $\Omega_i \cap \Omega$ such that $\Omega_i \cap \Gamma_D = \emptyset$, Lemma 3 gives the uniform Poincaré property if $\Omega_i \cap \Omega$ is convex and if there is $\rho > 0$ such that each patch contains a ball of radius $\rho \operatorname{diam}(\Omega_i)$ (and is trivially contained in a ball of radius $\operatorname{diam}(\Omega_i)$). Note that this is a reasonable restriction on the patches in view of condition (6).

Let us now turn to the patches close to the boundary where the Dirichlet conditions are prescribed. For simplicity, consider a two-dimensional setting, assume that the patches Ω_i are discs, and that $\Omega_i \cap \Gamma_D \neq \emptyset, \Omega_i \cap \Gamma_N = \emptyset$. Moreover, let $\Omega_i \cap \Gamma_D$ be a straight line segment. If $\Omega_i \cap \Omega$

is less than a half-disc (but $\Omega_i \cap \Omega$ still contains a ball with diameter ρ diam(Ω_i)), the reflection across Γ_D yields a convex domain $\tilde{\Omega} \subset \Omega_i$. For $u \in H^1(\Omega_i \cap \Omega)$ such that $u|_{\Gamma_D} = 0$, the antisymmetric extension across Γ_D gives an $H^1(\tilde{\Omega})$ function with zero average, and thus Lemma 3 gives

$$2\|u\|_{L^2(\Omega_i \cap \Omega)} = \|u\|_{L^2(\tilde{\Omega})} \leqslant \frac{\sqrt{2}}{\rho} \operatorname{diam}(\Omega_i) \|\nabla u\|_{L^2(\Omega_i \cap \Omega)}$$

The case that $\Omega_i \cap \Omega$ is bigger than a half-disc can be reduced to the above one by an appropriate mapping. The necessary condition for the Poincaré constant not to degenerate is that the length of the line segment $\Omega_i \cap \Gamma_D \geqslant \rho$ diam(Ω_i).

The case $\Omega_i \cap \Gamma_D \neq \emptyset$, $\Omega_i \cap \Gamma_N \neq \emptyset$ can be dealt with using similar ideas. Again, the necessary condition is that the length of the line segment $\Omega_i \cap \Gamma_D \geqslant \rho$ diam(Ω_i).

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