SIAM J. NUMER. ANAL.
Vol. 18, No. 3, June 1981

0036-1429/81/1803-0010 $01.00/0

THE p-VERSION OF THE FINITE ELEMENT METHOD*

I. BABUSKA†, B. A. SZABO‡ AND I. N. KATZ§

Abstract. In the p-version of the finite element method, the triangulation is fixed and the degree p, of the piecewise polynomial approximation, is progressively increased until some desired level of precision is reached.

In this paper, we first establish the basic approximation properties of some spaces of piecewise polynomials defined on a finite element triangulation. These properties lead to an a priori estimate of the asymptotic rate of convergence of the p-version. The estimate shows that the p-version gives results which are not worse than those obtained by the conventional finite element method (called the h-version, in which h represents the maximum diameter of the elements), when quasi-uniform triangulations are employed and the basis for comparison is the number of degrees of freedom. Furthermore, in the case of a singularity problem, we show (under conditions which are usually satisfied in practice) that the rate of convergence of the p-version is twice that of the h-version with quasi-uniform mesh. Inverse approximation theorems which determine the smoothness of a function based on the rate at which it is approximated by piecewise polynomials over a fixed triangulation are proved for both singular and nonsingular problems.

We present numerical examples which illustrate the effectiveness of the p-version for a simple one-dimensional problem and for two problems in two-dimensional elasticity. We also discuss roundoff error and computational costs associated with the p-version. Finally, we describe some important features, such as hierarchic basis functions, which have been utilized in COMET-X, an experimental computer implementation of the p-version.

CONTENTS

* Received by the editors June 18, 1979, and in revised form May 28, 1980.

† Institute of Physical Science and Technology and Department of Mathematics, University of Maryland, College Park, Maryland 20742. The work of this author was supported in part by the U.S. Department of Energy under contract E(40-1)3443.

‡ Greensfelder Professor of Civil Engineering, Washington University, St. Louis, Missouri 63130. The work of this author was supported in part by the Electric Power Research Institute under contract RP-1242-2.

§ Department of Mathematics and Systems Science, Washington University, St. Louis, Missouri 63130. The work of this author was supported in part by the U.S. Air Force Office of Scientific Research under grant AFOSR 77-3312.

1. Introduction. The finite element method, one of the most widely used numerical methods for solving certain types of differential equations, is based on approximating the solution by piecewise smooth functions, specifically polynomials, on convex subdomains such as triangles. In general, the degree of the polynomials is fixed at some arbitrarily chosen low number. No consensus exists at the present time concerning the most suitable (optimal) degree p of the polynomials.

The mathematical justification of the finite element method is based on asymptotic analyses in which p is kept bounded and the diameters of the element subdomains approach zero. However, it has been observed by several investigators that the sizes of elements used in practical computations are often outside the range of asymptotic behavior.

Because the maximum diameter of finite elements is usually denoted by h, we shall refer to this (conventional) approach as the *h-version* of the finite element method.

From the theoretical point of view, one can justify the finite element method, also in the asymptotic sense, when the subdomains are kept constant and the degree of the approximating polynomials tends to infinity. We shall refer to this method of approximation as the *p-version* of the finite element method.

The p-version of the finite element method is similar to the Ritz method, but there is one very important difference. In the p-version of the finite element method, the domain of interest is divided into convex subdomains and the polynomial approximants are piecewise smooth only over individual convex subdomains. In the Ritz method, on the other hand, the solution over the entire domain is approximated by smooth functions. This difference accounts for the greater versatility and higher rate of convergence of the p-version of the finite element method over both the Ritz method and the h-version of the finite element method, as demonstrated here.

In this paper, we analyze the p-version of the finite element method and its theory, and discuss the implementation characteristics of the method based on the computer program COMET-X, developed during the last few years at Washington University in St. Louis. We also examine the potential for further development of the p-version. We remark, from the computational point of view, and from the point of view of the architecture of the computer program, that there are significant differences between the p-version when p is in the range of 6, 7, 8 and the h-version when p is in the range 1, 2, 3.

We present a proof for the rate of convergence in the p-version and show that the polynomials are able to "absorb" singularities, including for example, corner singularities, when they are located at the vertices of triangles. This does not occur when the corner singularities are not located at vertices.

Comparison of the asymptotic behavior of the h-version, based on uniform or quasi-uniform mesh refinement on one hand, and the p-version on the other, the basis of comparison being the number of degrees of freedom, shows that the rate of convergence of the p-version cannot be slower than the rate of convergence of the h-version. Furthermore, when corner singularities are present at vertices, the rate of convergence of the p-version is exactly twice that of the h-version.

2. Basic notation. Throughout this paper, R^2 will be the two-dimensional Euclidean space $\{(x_1, x_2) = x | x_1, x_2 \in R\}$ and $\Omega \subset R^2$ will be a bounded domain with a piecewise smooth boundary $\partial\Omega$. In particular, we will deal with polygonal domains. (We exclude, for technical reasons, the slit domain, although the results of this paper can be generalized to this case with some, but not essential, technical difficulties.)

$\mathscr{E}(\bar{\Omega})$ shall be the space of all real C^∞ functions on Ω that allow continuous extensions of all derivatives to $\bar{\Omega}$. All functions in $\mathscr{E}(\bar{\Omega})$ that have compact support in Ω

form a subspace $\mathscr{D}(\Omega)\subset\mathscr{E}(\bar{\Omega})$. As usual, $L_2(\Omega)=H^0(\Omega)$ will be the space of all square-integrable functions on Ω with the following inner product:

$$(u,v)_{0,\Omega}=\int_\Omega uv\,dx,\quad \text{where } dx=dx_1\,dx_2,$$

and the corresponding norm $\|\cdot\|_{0,\Omega}$. In addition, for any integer $k\geqq 1$, the Sobolev spaces $H^k(\Omega)$ (respectively $H_0^k(\Omega)$) will be the completions of $\mathscr{E}(\bar{\Omega})$ (respectively $\mathscr{D}(\Omega)$) under the norm

$$\|u\|_{k,\Omega}^2=\sum_{0\leqq|\alpha|\leqq k}\|\mathscr{D}^\alpha u\|_{0,\Omega}^2,$$

where for each multi-integer (α_1,α_2), we have let $|\alpha|=\alpha_1+\alpha_2$ and

$$\mathscr{D}^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}.$$

The standard inner product in $H^k(\Omega)$ will be denoted by $(\cdot,\cdot)_{k,\Omega}$. For nonintegral $k>0$, the spaces $H^k(\Omega)$ and $H_0^k(\Omega)$ are defined by usual interpolation procedures. More precisely, for $k=k_0+\theta$, where $0<\theta<1$, we let $H^k=[H^{k_0},H^{k_0+1}]_{\theta,2}$ applying the usual K-method of interpolation. (For more information see [7].) (We note that $H^k=B_{2,2}^k$, where $B_{2,2}^k$ is the usual Besov space.)

For $\rho>0$, we let

$$Q(\rho)=\{(x_1,x_2)\,|\,|x_1|<\rho,\ |x_2|<\rho\},$$

$$\hat{Q}(\rho)=\{(x_1,x_2)\,|\,0<x_1<\rho,\ 0<x_2<\rho\},$$

and by $\mathscr{E}_{\mathrm{PER}}(\bar{Q}(\rho))\subset\mathscr{E}(\bar{Q}(\rho))$, we denote the space of all functions of period 2ρ on $\bar{Q}(\rho)$, and by $H_{\mathrm{PER}}^k(Q(\rho))$, its closure in $H^k(Q(\rho))$.

We will deal also with Sobolev spaces in one dimension. We let

$$I(\rho)=\{x_1\,|\,|x_1|<\rho\},$$

and we define $H^k(I(\rho))$, $H_0^k(I(\rho))$, $H_{\mathrm{PER}}^k(I(\rho))$ as above.

We also need to introduce the space $\mathscr{P}_p(\Omega)\subset\mathscr{E}(\bar{\Omega})$ of all algebraic polynomials of degree not higher than p and $\mathscr{F}_p(Q(\rho))$ (and $\mathscr{F}_p(I(\rho))$, the space of all trigonometric polynomials of degree at most p (and period 2ρ).

3. The concept of p-convergence of the finite element method.

3.1. The model problem. We will be interested in the following model problem:

$$-\Delta u+u=f\quad\text{on }\Omega_0,\qquad f\in H^0(\Omega_0), \tag{3.1}$$

$$\Gamma u=0\quad\text{on }\partial\Omega_0, \tag{3.2}$$

where Ω_0 is a bounded polygonal domain and $\Gamma u=u$ or $\Gamma u=\partial u/\partial n$. We can easily generalize our results to other boundary conditions. As usual, we will interpret the problem (3.1), (3.2) in a weak sense; namely, we seek a $u_0\in H_0^1(\Omega_0)$ (respectively $u_0\in H^1(\Omega_0)$) so that

$$B(u_0,v)=(f,v)_{0,\Omega_0}\quad\text{for all } v\in H_0^1(\Omega_0)\text{ (respectively } v\in H^1(\Omega_0)), \tag{3.3}$$

where we have used the notation

$$B(u_0,v)=(u_0,v)_{1,\Omega_0}. \tag{3.4}$$

A u_0 satisfying (3.3) obviously exists and is uniquely determined.

3.2. A description of the p-version of the finite element method. Let $\mathcal{S}$ be a (fixed) triangularization of Ω_0. Thus $\mathcal{S}=\{T_i\}$ for $i=1,\cdots,m$, where each T_i is an (open) triangle, where $\bigcup_{i=1}^m \bar{T}_i=\bar{\Omega}_0$ and where each pair $\bar{T}_i$, $\bar{T}_j$, $i\neq j$, have either a common (entire) side or a common vertex or are disjoint; that is, $\bar{T}_i\cap\bar{T}_j=\emptyset$. Let $\mathcal{P}_p^{[\mathcal{S}]}(\Omega_0)\subset H^1(\Omega_0)$ denote the subset of all functions $u\in H^1(\Omega_0)$ such that if $u_{(T_i)}$ is the restriction of u on T_i, then we have $u_{(T_i)}\in\mathcal{P}_p(T_i)$; that is, $\mathcal{P}_p^{[\mathcal{S}]}(\Omega_0)$ consists of all functions in $H^1(\Omega_0)$ which are piecewise polynomials of degree at most p. Furthermore, let $\mathcal{P}_{p,0}^{[\mathcal{S}]}(\Omega_0)=\mathcal{P}_p^{[\mathcal{S}]}(\Omega_0)\cap H_0^1(\Omega_0)$.

The concept of the p-version of the finite element method consists of finding a $u_p\in\mathcal{P}_{p,0}^{[\mathcal{S}]}(\Omega_0)$ (if the Dirichlet problem is under consideration) or a $u_p\in\mathcal{P}_p^{[\mathcal{S}]}(\Omega_0)$ (if the Neumann problem is under consideration), for each $p=1,2,\cdots$, so that (3.3) holds for all v in the appropriate space.

Study of the p-version of the finite element method was initiated at the School of Engineering and Applied Science of Washington University in St. Louis [25] in 1970. It has been implemented there to study various aspects of stress analysis and has shown very good results, particularly in connection with linear elastic fracture mechanics. Development of the p-version is continuing at the Center for Computational Mechanics at Washington University.

3.3. The basic approximation properties of $\mathcal{P}_p^{[\mathcal{S}]}(\Omega_0)$ and $\mathcal{P}_{p,0}^{[\mathcal{S}]}(\Omega_0)$.

THEOREM 3.1. *Let $u\in H^k(\Omega_0)$. Then there exists a sequence $z_p\in\mathcal{P}_p^{[\mathcal{S}]}(\Omega_0)$, $p=1,2,\cdots$, such that for any $0\leqq l\leqq k$ (l, k not necessarily integral),*

$$\|u-z_p\|_{l,\Omega_0}\leqq Cp^{-(k-l)}\|u\|_{k,\Omega_0}, \tag{3.5}$$

where C is independent of u and p. (C depends, for example, on l and k.)

Proof. The proof is standard. We first prove (3.5) for integral l and k. We will construct a specific $z_p\in\mathcal{P}_p(\Omega_0)$ so that (3.5) is satisfied.

Choose $\rho_0>0$ so that $\Omega_0\subset Q(\rho_0)$. Because Ω_0 is a polygon, it is a Lipschitz domain and thus, there exists an extension $U\in H^k(Q(2\rho_0))$ of u such that supp $U\subset Q(\frac{3}{2}\rho_0)$ and such that

$$\|U\|_{k,Q(2\rho_0)}\leqq C\|u\|_{k,\Omega_0}, \tag{3.6}$$

where C is independent of u. As usual, we have that $U=Tu$, where T is a linear mapping of $H^0(\Omega_0)$ into $H^0(Q(2\rho_0))$ (see, for example, [24]), which also maps $H^k(\Omega_0)$ into $H^k(Q(2\rho_0))$.

Now let ϕ be a (one-to-one) mapping of $Q(\pi/2)$ onto $Q(2\rho_0)$ defined by the following:

$$x=\Phi(\xi)=2\rho_0(\sin\xi_1,\sin\xi_2). \tag{3.7}$$

Let

$$V(\xi)=U(\Phi(\xi)),$$

and let

$$\tilde{Q}=\Phi^{[-1]}\left[Q\left(\frac{3}{2}\rho_0\right)\right]\subset Q\left(\frac{\pi}{2}\right) \tag{3.8}$$

where $\Phi^{[-1]}$ denotes the inverse of Φ. We note that

$$\operatorname{supp} V\subset\tilde{Q}. \tag{3.9}$$

Obviously, Φ is a regular, analytic, one-to-one mapping of $\tilde{Q}$ onto $Q(\frac{3}{2}\rho_0)$. If we regard

Φ as a map from $Q(\pi)$ onto $Q(2\rho_0)$, $V=U(\Phi)\in H^k_{\text{per}}(Q(\pi))$ is symmetric with respect to the lines $\xi_i=\pm\pi/2$ for $i=1$ or 2 and (3.6) and (3.9) show that

$$\|V\|_{k,Q(\pi)}\leqq C\|u\|_{k,\Omega_0}. \tag{3.10}$$

It is well known that the partial sum t_p of the Fourier series of V gives a sequence of trigonometric polynomials $t_p\in\mathcal{F}_p(Q(\pi))$ which satisfy the following, for integer $k\geqq l\geqq 0$:

$$\begin{aligned}\|V-t_p\|_{l,Q(\pi)}&\leqq Cp^{-(k-l)}\|V\|_{k,Q(\pi)}\\&\leqq Cp^{-(k-l)}\|u\|_{k,\Omega_0}.\end{aligned} \tag{3.11}$$

Each t_p has the same symmetry as V with respect to the lines $\xi_i=\pm\pi/2$, for $i=1$ or 2. Thus, it is readily seen that $t_p(\xi)=z_p(\Phi(\xi))$, where z_p is an algebraic polynomial of degree not higher than p. Because Φ is a regular, analytic map of $\tilde{Q}$ onto $Q(\frac{3}{2}\rho_0)$, (3.11) yields (3.5) for integer l and k.

We now generalize our result to nonintegral l and k. Recall that for a given (fixed) p, the polynomial z_p was constructed using a linear map L_p. In fact, $L_pu=z_p$, where L_p is a linear map of $H^0(\Omega_0)$ into $\mathcal{P}_p(\Omega_0)$ that satisfies (3.5) for integer l and k. If we now apply general interpolation theory, we obtain (3.5) for all $0\leqq l\leqq k$.

The proof of the next theorem is more complicated.

THEOREM 3.2. *Let* $u\in H^k(\Omega_0)\cap H^1_0(\Omega_0)$. *Then there exists a sequence* $z_p\in\mathcal{P}^{[\mathcal{S}]}_{p,0}(\Omega_0)$, *for* $p=1,2,\cdots$, *such that for any* $k>1$ (*not necessarily integral*) *and any* $\varepsilon>0$, *we have*

$$\|u-z_p\|_{1,\Omega_0}\leqq Cp^{-(k-1)+\varepsilon}\|u\|_{k,\Omega_0}, \tag{3.12}$$

where C *is independent of* p *and* u. (C *depends, for example on* ε *and* k.)

Remark 1. In contrast to Theorem 3.1, Theorem 3.2 is false if $\mathcal{P}_{p,0}(\Omega_0)$ is considered instead of $\mathcal{P}^{[\mathcal{S}]}_{p,0}(\Omega_0)$. This is easy to see if Ω_0 is, for example, the L-shaped domain shown in Fig. 3.1.

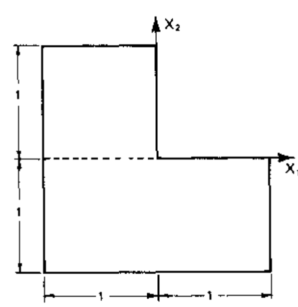

FIG. 3.1. *An L-shaped domain.*

In fact, any $u\in\mathcal{P}_{p,0}(\Omega_0)$ is zero on $\{(x_1,0)|0<x_1<1\}$ and therefore, because it is a polynomial, it has to be zero on the entire set $\{(x_1,0)|-1<x_1<1\}$. This, of course, leads to a contradiction because of Sobolev's embedding theorem of $H^1_0(\Omega_0)$ into $H^0(I(1))$.

Remark 2. It is not clear whether the term ε in (3.12) can be removed.

Remark 3. The theorem can be stated more generally. We have restricted ourselves to proving estimates in the $\|\cdot\|_{1,\Omega_0}$-norm only because it is sufficient for our purpose.

We will state a lemma before proving Theorem 3.2.

LEMMA 3.1. *Let S be a triangle with vertices A_i, $i = 1, 2, 3$, and sides s_i, $i = 1, 2, 3$, as in Fig.* 3.2.

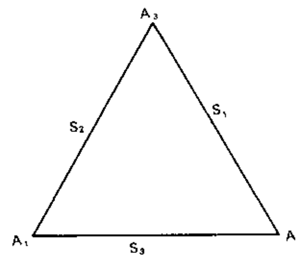

FIG. 3.2. *The typical triangle.*

Let $v \in \mathscr{P}_{p,0}(s_1)$. Then there exists a $V \in \mathscr{P}_p(S)$ such that $V = 0$ on s_2 and s_3, $V = v$ on s_1 and

$$\|V\|_{1,S} \leqq C\|v\|_{1,s_1}, \tag{3.13}$$

where C (dependent on S) is independent of v and p.

Remark. By $v \in \mathscr{P}_{p,0}(s_1)$ we mean, of course, a polynomial in a single variable on s_1 that vanishes at the end points of s_1, i.e., at the vertices A_2, A_3.

Proof. Without any loss of generality, we can assume that S is the triangle shown in Fig. 3.3 with vertices (0, 0), (1, 0), (1, 1).

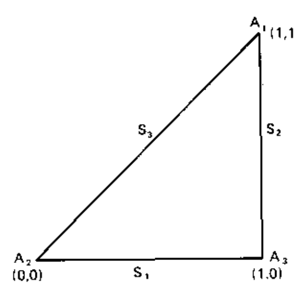

FIG. 3.3. *The standard triangle.*

Then, $s_1 = \{(x_1, 0) | 0 < x_1 < 1\}$. Because $v(x)$ is a polynomial and because $v(0) = v(1) = 0$ by assumption, we know that

$$v(x_1) = x_1(1 - x_1)v_1(x_1),$$

where $v_1(x_1)$ is a polynomial of degree at most $p - 2$. Let

$$V(x) = V(x_1, x_2) = v(x_1)\frac{(x_1 - x_2)}{x_1}. \tag{3.14}$$

Obviously, $V \in \mathscr{P}_p(S)$, $V = 0$ on s_2 and s_3, and $V = v$ on s_1. Since x_2/x_1 is bounded on S, we find that $\|V\|_{0,S} \leqq \|v\|_{1,s_1}$. Moreover, since

$$\int_S \left(\frac{\partial V}{\partial x_i}\right)^2 dx_1\, dx_2 = \int_0^1 dx_2 \int_{x_2}^1 \left(\frac{\partial V}{\partial x_i}\right)^2 dx_1,$$

we obtain (3.13) upon observing that $v^2(x_1) \leqq x_1\|v\|^2_{1,s_1}$.

We can now prove Theorem 3.2.

Proof of Theorem 3.2.

1) Let $k > 2$. By Theorem 3.1, there exists a $z_p \in \mathscr{P}_p(\Omega_0)$ such that

$$\|u - z_p\|_{l,\Omega_0} \leqq Cp^{-(k-l)}\|u\|_{k,\Omega_0} \quad \text{for } l \leqq k. \tag{3.15}$$

Let $\{x^{[j]}\}$, $j = 1, \cdots, \hat{m}$, be the set of all the vertices of the triangles $T_i \in \mathscr{S}$ belonging to $\partial\Omega_0$. Because $H^{1+\varepsilon}(\Omega_0)$, where $\varepsilon > 0$, is embedded in the space of continuous functions, we can obviously modify z_p by subtracting a polynomial $\tilde{z}$ of fixed degree $p_0 \leqq \hat{m}$ and obtain a $z_p^* \in \mathscr{P}_p(\Omega_0)$ that vanishes at the points $\{x^{[j]}\}$ and satisfies the following:

$$\|u - z_p^*\|_{l,\Omega_0} \leqq C(p^{-(k-l)} + p^{-(k-1-\varepsilon)})\|u\|_{k,\Omega_0}, \tag{3.16}$$

where $\varepsilon > 0$ is arbitrary. Of course, $\hat{z} = z_p - z_p^*$ is a polynomial of fixed (independent of p) degree p_0 determined by the values $\{z_p(x^{[j]})\}$. Since Theorem 3.1 shows that $|z_p(x^{[j]})| \leqq Cp^{-(k-1)+\varepsilon}\|u\|_{k,\Omega_0}$, it follows that $\|\tilde{z}\|_{l,\Omega_0} \leqq Cp^{-(k-1)+\varepsilon}$ for all $0 \leqq l \leqq \hat{m}$. Then, because $\|\tilde{z}\|_{r,\Omega_0} = \|\tilde{z}\|_{\hat{m},\Omega_0}$ for any $r \geqq \hat{m}$, (3.16) follows readily.

Since $u = 0$ on every side $s \subset \partial\Omega_0$ of some $T \in \mathscr{S}$, we have $\|z_p^*\|_{1,s} \leqq \|u - z_p^*\|_{2,\Omega_0}$, by applying the Sobolev imbedding theorem. Thus, we can use Lemma 3.1 to find a $z_p^{**} \in \mathscr{P}_p^{[\mathscr{S}]}(\Omega_0)$ such that $z_p^0 = z_p^* - z_p^{**} \in \mathscr{P}_{p,0}^{[\mathscr{S}]}(\Omega_0)$ and

$$\|u - z_p^0\|_{1,\Omega_0} \leqq C\|u - z_p^*\|_{2,\Omega_0} \leqq Cp^{-(k-2)}\|u\|_{k,\Omega_0}, \tag{3.17}$$

where C depends on $k > 2$ but is independent of u. Note that we can rewrite (3.17) as follows:

$$\|u - z_p^0\|_{1,\Omega_0} \leqq Cp^{-(k-1)(1-1/(k-1))}\|u\|_{k,\Omega_0}. \tag{3.18}$$

2) Now let R_p be the orthogonal projection in the scalar product of H^1 of

$$H^k(\Omega_0) \cap H_0^1(\Omega_0) \quad \text{onto } \mathscr{P}_{p,0}^{[\mathscr{S}]}(\Omega_0),$$

and let $z_p = R_p u$. We obviously have

$$\|z_p - u\|_{1,\Omega_0} \leqq \|u\|_{1,\Omega_0}, \tag{3.19}$$

and, from (3.18), we have the following for $k > 2$:

$$\|z_p - u\|_{1,\Omega_0} \leqq C(k)p^{-(k-1)(1-1/(k-1))}\|u\|_{k,\Omega_0}. \tag{3.20}$$

Now let

$$\tilde{H}^s(\Omega_0) = [H_0^1, H^k(\Omega_0) \cap H_0^1]_{(s-1)/(k-1),2} \quad \text{for } 1 < s < k. \tag{3.21}$$

Standard interpolation results imply that the following holds for $1 < s < k$:

$$\|z_p - u\|_{1,\Omega_0} \leqq C(k, s)p^{-\mu}\|u\|_{\tilde{H}^s(\Omega_0)} \quad \text{for all } u \in \tilde{H}^s(\Omega_0), \tag{3.22}$$

where

$$\mu = (k-1)\left(1 - \frac{1}{k-1}\right)\left(\frac{s-1}{k-1}\right) = (s-1)\left(1 - \frac{1}{k-1}\right). \tag{3.23}$$

Thus, given some $\varepsilon > 0$ and some $s > 1$, we can select a $k_0 > 1$ so that

$$(s-1)\left(1 - \frac{1}{k_0 - 1}\right) \geqq (s-1) - \varepsilon, \tag{3.24}$$

and obtain (3.12), at least with $\|u\|_{\tilde{H}^k(\Omega_0)}$ replacing $\|u\|_{k,\Omega_0}$. But since the spaces $\tilde{H}^k(\Omega_0)$ and $H^k(\Omega_0) \cap H_0^1$ are equivalent when Ω_0 is a polygonal domain (see [3], also [29]), the proof is complete.

3.4. Inverse approximation theorems. We have proven theorems about the approximability properties of the following spaces:

$$\mathcal{P}_p^{[\mathcal{S}]}(\Omega_0) \quad \text{and} \quad \mathcal{P}_{p,0}^{[\mathcal{S}]}(\Omega_0).$$

Now we will prove some inverse approximation theorems.

THEOREM 3.3. *Let $v \in H^k(Q(\rho))$, where $k \geqq 0$ is integral and suppose there exists a sequence of polynomials $z_p \in \mathcal{P}_p(Q(\rho))$, $p = 1, 2, \cdots$, such that*

$$\|v - z_p\|_{k,Q(\rho)} \leqq \frac{K}{p^r} \quad \text{for } r > 0. \tag{3.25}$$

Then $v \in H^{k+r-\varepsilon}(Q(\rho^*))$, for any $0<\rho^*<\rho$ and $\varepsilon>0$ and the restriction of v onto $Q(\rho^*)$, satisfies the following:

$$\|V\|_{k+r-\varepsilon,Q(\rho^*)} \leqq A(\rho, \rho^*, k, r, \varepsilon)[\|v\|_{k,Q(\rho)} + K]. \tag{3.26}$$

Proof. Let $\omega = (x_1^2 - \rho^2)^{k+2}(x_2^2 - \rho^2)^{k+2}$ and set $v^* = v\omega$, $z^*_{p+4(k+2)} = z_p\omega$, $V^*(\xi) = v^*(\Phi(\xi))$ and

$$t^*_{p+4(k+2)}(\xi) = z^*_{p+4(k+2)}(\Phi(\xi)) \in \mathcal{F}_{p+4(k+2)}(Q(\pi)),$$

where

$$\Phi(\xi) = x = \rho(\sin \xi_1, \sin \xi_2).$$

Since

$$\|V^* - t^*_{p+4k+2}(\xi)\|_{k,Q(\pi)} \leqq \frac{CK}{p^r}, \tag{3.27}$$

it follows from [17, Theorem (5.4.1)] that

$$\|V^*\|_{k+r-\varepsilon,Q(\pi)} \leqq A[\|V^*\|_{k,Q(\pi)} + K].$$

(For a direct proof using interpolation theory, see, for example [3].) (In fact, the aforementioned theorem provides an estimate for the norm of V^* in the Nikolsky space $B^{k+r}_{2,\infty}(Q(\pi))$. This norm majorizes the norm $\|\cdot\|_{k+r-\varepsilon,Q(\pi)}$.)

Now, using the facts that the mapping Φ is one-to-one analytic on $Q(\rho^*)$, and $\omega(x) > \alpha > 0$ for some $\alpha > 0$ on $Q(\rho^*)$, we immediately obtain (3.26). Inequality (3.25) holds only on $Q(\rho^*)$ and is not in general true on $Q(\rho)$. Nevertheless, we can prove the next theorem.

THEOREM 3.4. *Let $v \in H^k(Q(\rho))$ and suppose that the remaining assumptions of Theorem 3.3 are satisfied. Then $v \in H^{k+r/2-\varepsilon}(Q(\rho))$ for any $\varepsilon > 0$ and*

$$\|v\|_{k+r/2-\varepsilon,Q(\rho)} \leqq A(\varepsilon)[\|v\|_{k,Q(\rho)} + K]. \tag{3.28}$$

The proof of this theorem is again a consequence of Theorem 5.4.1 of [17], provided that the following inequality of Bernstein type holds for each integer $k \geqq 0$:

$$\|z_p\|_{k,Q(\rho)} \leqq Cp^{2k}\|z_p\|_{0,Q(\rho)}, \tag{3.29}$$

for all $z_p \in \mathcal{P}_p(Q(p))$, where C is independent of p and z_p.

Let us remark that in the case of trigonometric polynomials, (3.29) holds with p^k replacing p^{2k}. We will prove (3.29) in the next few lemmas.

LEMMA 3.2. *Let $z_p(x)$ be a polynomial of degree p in one variable. Then*

$$\left\|\frac{d^s z_p}{dx^s}\right\|_{0,I} \leqq C(s)p^{2s}\|z_p\|_{0,I}. \tag{3.30}$$

Proof. By Schmidt's inequality, we have

$$\int_{-1}^{+1} f'(x)^2\,dx \leqq \frac{(N+1)^4}{2}\int_{-1}^{+1} f^2(x)\,dx, \tag{3.31}$$

for polynomials f of degree not greater than N (see [6]). The result is now easily obtained.

LEMMA 3.3. *Let $z_p(x)\in\mathcal{P}_p(Q(1))$. Then for each integer $k\geqq 0$ we have*

$$\|z_p\|_{k,Q(1)}\leqq C(k)p^{2k}\|z_p\|_{0,Q(1)}. \tag{3.32}$$

Proof. For each fixed x_2, we have $z_p(x_1,x_2)\in\mathcal{P}_p(I)$. Thus, Lemma 3.2 implies that

$$\int_{-1}^{+1}\left[\frac{\partial z_p}{\partial x_1}(x_1,x_2)\right]^2 dx_1\leqq Cp^4\int_{-1}^{+1} z_p^2(x_1,x_2)\,dx_1. \tag{3.33}$$

If we now integrate (3.33) with respect to x_2, we find that

$$\left\|\frac{\partial z_p}{\partial x_1}\right\|_{0,Q(1)}\leqq Cp^2\|z_p\|_{0,Q(1)}. \tag{3.34}$$

Since a similar estimate holds for $\partial z_p/\partial x_2$, (3.32) follows by induction. Since (3.32) is equivalent to (3.29), Theorem 3.4 is now proven.

3.5. Convergence results for the p-version of the finite element method. Theorems 3.1 and 3.2 lead immediately to the following a priori estimate of the rate of convergence of the p-version of the finite element method.

THEOREM 3.5. *Let $u_0\in H^k(\Omega_0)$, where $k>1$, be the solution of the problem* (3.1) *and* (3.2) *and let u_p be the finite element approximation. Then*

$$\|u_0-u_p\|_{1,\Omega_0}\leqq C(k,\varepsilon)p^{-(k-1)+\varepsilon}\|u_0\|_{k,\Omega}, \tag{3.35}$$

for any $\varepsilon>0$. If the Neumann boundary conditions are under consideration (that is, $\Gamma=\partial u/\partial n$), ε can be taken to be zero.

A polynomial of degree p has N degrees of freedom, where $N\approx p^2$. Thus, $\mathcal{P}_p^{[\mathcal{S}]}$ (and $\mathcal{P}_{p,0}^{[\mathcal{S}]}$) is of dimension N, where $N\approx p^2$, and hence (3.35) can be rewritten in the following form:

$$\|u_0-u_p\|_{1,\Omega_0}\leqq C(k,\varepsilon)N^{-(k-1)/2+\varepsilon}\|u_0\|_{k,\Omega_0}. \tag{3.36}$$

For the conventional finite element method (h-version) on a quasi-uniform mesh, we have

$$\|u_0-u_h\|_{1,\Omega_0}\leqq Ch^{\mu}\|u_0\|_{k,\Omega_0}, \tag{3.37}$$

where $\mu=\min(k-1,q)$ and where q is the degree of the complete polynomial used in the elements. Since in this case, the number of degrees of freedom N satisfies the relation $N\approx h^{-2}$, we can rewrite (3.37) in the following form:

$$\|u_0-u_h\|_{1,\Omega_0}\leqq CN^{-\mu/2}\|u_0\|_{k,\Omega_0}. \tag{3.38}$$

Note that the convergence rate expressed by (3.38) is optimal (up to, perhaps, an arbitrarily small $\varepsilon>0$; see [3]). Thus, the p-version gives results which are (neglecting a possible $\varepsilon>0$) not worse than the conventional h-version on a quasi-uniform mesh if we are comparing the number of degrees of freedom that are required to obtain a certain accuracy. In addition, the p-version convergence can be much better as there is no restriction on its convergence rate due to an upper bound on the degree of the polynomials.

We will see in the next section (see Theorem 4.3) that, under some conditions which are usually satisfied in practice, the factor $\frac{1}{2}$ can be removed from the exponent in (3.36). Thus, under these conditions, the p-version will be superior to the usual h-version of the finite element method on a quasi-uniform mesh. However, let us remark that when the usual h-version of the finite element method is used with the proper refinement of elements, then the convergence rate of the h-version can be better than that of the p-version on a fixed mesh (see [3]). Although the general theory for a method combining the h and p versions is not yet developed, we can expect that the theoretical and practical advantages of both approaches can be combined.

Let us now assume that the convergence rate of the p-version of the finite element method is some $r>0$; that is, assume that

$$\|u_0-u_p\|_{1,\Omega_0}\leqq Kp^{-r}. \tag{3.39}$$

Then the following theorem holds:

THEOREM 3.6. *Let* $u_0\in H^1(\Omega_0)$ *and suppose that* (3.39) *holds. Then*:

(i) $u_0\in H^{1+r-\varepsilon}(\Omega^*)$ *where* Ω^* *is any domain such that* $\bar{\Omega}^*\subset T_i$ *for some* $i=1,\cdots,m$, *where* T_i *are the triangles of the triangulation* $\mathcal{S}$ *and*

$$\|u_0\|_{1+r-\varepsilon,\Omega^*}\leqq A(\Omega^*,r,\varepsilon)(\|u_0\|_{1,\Omega_0}+K). \tag{3.40}$$

(ii) $u_0\in H^{1+r/2-\varepsilon}(T_i)$ *for each* $i=1,\cdots,m$ *and*

$$\|u_0\|_{1+r/2-\varepsilon,T_i}\leqq A(T_i,r,\varepsilon)(\|u_0\|_{1,\Omega_0}+K). \tag{3.41}$$

Proof. Theorems 3.3 and 3.4 are obviously valid not only for a rectangle Q but for any parallelogram.

(i) From Theorem 3.3, we see that (3.40) holds for any Ω^*, that is, of the form of a parallelogram. It is now an easy matter to obtain (3.40).

(ii) Since any T_i can be covered (with overlapping) by a finite set of parallelograms, (3.41) follows directly from Theorem 3.4.

The practical importance of Theorem 3.6 lies in the observation that the triangulation of Ω has to be constructed so that possible singularities of u_0 are located on the boundaries of the triangles. This was done exactly in the linear elastic fracture mechanics problems analyzed by Szabo and Mehta [26] using the p-version of the finite element method.

4. The singularity problem and the p-version of the finite element method.

4.1. Preliminaries. In this section, we will write $\hat{Q}$ instead of $\hat{Q}(1)$. $\hat{Q}(\rho)$ was defined in § 2. Let T_1 be an open triangle with a vertex at the origin and suppose that $\bar{T}_1\subset\hat{Q}(1/3)\cup(0,0)$ (see Fig. 4.1).

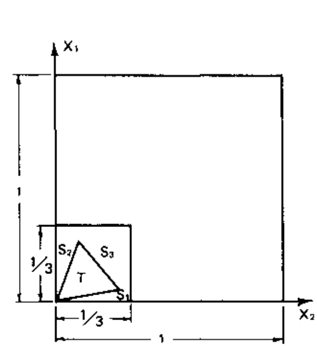

FIG. 4.1. *Triangle with a vertex at a singularity.*

We will denote the sides of T meeting the origin by s_1 and s_2 and the remaining one by s_3.

We now define a mapping $\hat{\Phi}$ of $\hat{Q}(\pi/2)$ onto $\hat{Q}$. Let

$$\hat{\Phi}(\xi) = x = (\sin^2 \xi_1, \sin^2 \xi_2) \quad \text{for all } \xi \equiv (\xi_1, \xi_2) \in \hat{Q}\left(\frac{\pi}{2}\right).$$

Let $\hat{\Phi}^{-1}$ denote the inverse of $\hat{\Phi}$ and let

$$T^{\Phi} = \hat{\Phi}^{-1}(T), \qquad s_i^{\Phi} = \hat{\Phi}^{-1}(s_i) \quad \text{for } i = 1, 2 \text{ or } 3.$$

T^{Φ} is a curvilinear triangle with smooth sides and positive angles. In fact, the line $x_2 = cx_1 (0 < c < \infty)$ will be mapped into $\sin^2 \xi_2 = c \sin^2 \xi_1$, or

$$\xi_2 = \arcsin c^{1/2} \sin \xi_1.$$

Thus, T^{Φ} is a curvilinear triangle and $\bar{T}^{\Phi} \subset \hat{Q}(\rho_0) \cup (0, 0)$ where $\rho_0 = \arcsin (1/\sqrt{3})$. It can be seen easily that $\sin 2\xi_1/\sin 2\xi_2$ is a function bounded above and below on T^{Φ}.

Now let $v \in H^1(T)$ be given and set

$$V(\xi) = v(\hat{\Phi}(\xi)).$$

We have the following:

LEMMA 4.1. *Let $v \in H^1(T)$. Then $V \in H^1(T^{\Phi})$ and*

$$c_1 \|v\|_{1,T} \leqq \|V\|_{1,T^{\Phi}} \leqq c_2 \|v\|_{1,T}, \tag{4.1}$$

where $0 < c_1 < c_2 < \infty$ are independent of v.

Proof. We will first show that

$$c_1 \left\| \frac{\partial v}{\partial x_i} \right\|_{0,T} \leqq \left\| \frac{\partial V}{\partial \xi_i} \right\|_{0,T^{\Phi}} \leqq c_2 \left\| \frac{\partial v}{\partial x_i} \right\|_{0,T} \quad \text{for } i = 1 \text{ or } 2. \tag{4.2}$$

We have

$$\frac{\partial V}{\partial \xi_i}(\xi) = \frac{\partial v}{\partial x_i} \frac{\partial x_i}{\partial \xi_i} = \frac{\partial v}{\partial x_i}(\hat{\Phi}(\xi)) \sin 2\xi_1.$$

Therefore,

$$\int_{T^{\Phi}} \left(\frac{\partial V}{\partial \xi_i}\right)^2 d\xi_1\, d\xi_2 = \int_T \left(\frac{\partial v}{\partial x_i}\right)^2 \sin^2 2\xi_1 \frac{dx_1}{\sin 2\xi_1} \frac{dx_2}{\sin 2\xi_2}.$$

Since $\sin 2\xi_1/\sin 2\xi_2$ is a function bounded from above and below on T^{Φ}, we obtain (4.2). We now observe that if $1 \leqq p, q < \infty$ and $1/p + 1/q = 1$, then

$$\begin{aligned} \int_{T^{\Phi}} V^2\, d\xi &= \int_T v^2 \frac{dx_1}{\sin 2\xi_1} \frac{dx_2}{\sin 2\xi_2} \\ &\leqq \left[\int_T v^{2p}\, dx\right]^{1/p} \left[\int_T \left(\frac{1}{\sin 2\xi_1} \frac{1}{\sin 2\xi_2}\right)^q dx\right]^{1/q}. \end{aligned} \tag{4.3}$$

Note that since $\xi_i \approx x_i^{1/2}$ in a neighborhood of the origin, we have $1/\sin 2\xi_1 \approx x_i^{-1/2}$. Therefore, the second term in (4.3) is bounded if $p = 3$ and $q = 3/2$. On the other hand, by the Sobolev embedding theorem, we have

$$\left| \int_T v^6\, dx \right|^{1/3} \leqq C \|v\|_{1,T}^2.$$

Thus, it follows that

$$\left[\int_{T^\Phi} V^2\, d\xi\right]^{1/2} \leqq C\|v\|_{1,T}. \tag{4.4}$$

This estimate and (4.2) show that

$$\|V\|_{1,T^\Phi} \leqq C\|v\|_{1,T}.$$

Since (4.3) also implies the following,

$$\|v\|_{0,T} \leqq \|V\|_{0,T^\Phi},$$

we can easily complete the proof of the lemma.

LEMMA 4.2. *Suppose that $v(x)$ is a function of one variable defined for $0<x<1$ which satisfies the following*:

$$\int_0^1 v^2 x^{-1}\, dx + \int_0^1 \left(\frac{dv}{dx}\right)^2 x\, dx \leqq A^2 < \infty. \tag{4.5}$$

Let S be the triangle with vertices (0, 0), (1, 0), (1, 1) (*as in Fig.* 3.3) *and let*

$$u(x_1, x_2) = v(x_1)\left(1 - \frac{x_2}{x_1}\right).$$

Then

$$\|u\|_{1,S} \leqq CA, \tag{4.6}$$

where C is independent of v.

Proof. We have

$$\begin{aligned}\int_S u^2\, dx &= \int_0^1 v^2(x_1)\, dx_1 \int_0^{x_1} \left(1 - \frac{x_2}{x_1}\right)^2 dx_2 \\ &= \frac{1}{3}\int_0^1 v^2(x_1)x_1\, dx_1 \leqq CA^2.\end{aligned} \tag{4.7}$$

Furthermore,

$$\begin{aligned}\int_S \left(\frac{\partial u}{\partial x_1}\right)^2 dx &\leqq 2\left[\int_0^1 \left(\frac{\partial v}{\partial x_1}\right)^2 dx_1 \int_0^{x_1}\left(1-\frac{x_2}{x_1}\right)^2 dx_2 + \int_0^1 v^2\, dx_1 \int_0^{x_1} \frac{x_2^2}{x_1^4}\, dx_2\right] \\ &\leqq \frac{2}{3}\left[\int_0^1 x_1\left(\frac{\partial v}{\partial x_1}\right)^2 dx_1 + \int_0^1 x_1^{-1} v^2\, dx_1\right] \leqq CA^2,\end{aligned} \tag{4.8}$$

$$\int_S \left(\frac{\partial u}{\partial x_2}\right)^2 dx \leqq \int_0^1 v^2\, dx_1 \int_0^{x_1} \frac{1}{x_1^2}\, dx_2 \leqq \int_0^1 v^2 x_1^{-1}\, dx_1 \leqq A^2. \tag{4.9}$$

Since (4.7), (4.8) and (4.9) give (4.6), the lemma is proven.

Remark. If v is a smooth function, (4.5) implies that $v(0)=0$. Moreover, if v is a polynomial, then u is a polynomial (in two variables) as well.

4.2. Approximation properties of the space $\mathscr{P}_p(T)$. Suppose that $\psi_\gamma = \{u_\Delta | 0 < \Delta < \Delta_0\}$ is a one-parameter family of functions on $\hat{O}$ that satisfies the following, where Δ_0, γ and α are certain nonnegative constants:

(i) $u_\Delta \in \mathscr{E}(\hat{Q})$ (*not* $\mathscr{E}(\bar{\hat{Q}})$) for all $0<\Delta<\Delta_0$,

(ii) supp $u_\Delta \subset R_\alpha$ for all $0<\Delta<\Delta_0$, where

$$R_\alpha \equiv \left\{x \in \hat{Q}\left(\frac{1}{3}\right), \frac{x_1}{\alpha} < x_2 < \alpha x_1\right\},$$

(iii) there is a nonnegative, nondecreasing function $C(\cdot)$ so that

$$|D^\alpha u_\Delta(x)| \leqq C(|\alpha|)(\max(]x[, \Delta))^{-\{|\alpha|-\gamma\}},$$

for all $0<\Delta<\Delta_0$ and all $x \in \hat{Q}$, where $]x[= \min(x_1, x_2)$ and $\{a\} = \max(a, 0)$.

We shall study how polynomials in $\mathscr{P}_p(T)$ approximate functions chosen from such families, when $T \subset \hat{Q}$.

Let

$$U_\Delta(\xi) = u_\Delta(\hat{\Phi}(\xi)) \quad \text{for } \xi \in \hat{Q}\left(\frac{\pi}{2}\right),$$

and let

$$\psi_\gamma^\Phi \equiv \{U_\Delta(\xi) | 0<\Delta<\Delta_0\}.$$

We now prove the following theorem:

THEOREM 4.1. *Let $k \geqq 0$. Then*

$$U_\Delta \in H^k\left(\hat{Q}\left(\frac{\pi}{2}\right)\right) \quad \textit{for all } 0<\Delta<\Delta_0,$$

and

$$\|U_\Delta\|_{k,Q(\pi/2)} \leqq C(k)\Delta^{-\{1/2\{k-2\gamma\}-1/2\}}, \tag{4.10}$$

where C is independent of Δ.

We first prove some auxiliary lemmas.

LEMMA 4.3. *For $0<t<\pi/2$, $n \geqq 1$ and $1 \leqq k \leqq n$ where k and n are integers, let*

$$^nW_k(t) = \sum_{j=1}^{k} (-1)^{k-j}\binom{k}{j} \sin^{2(k-j)} t \frac{d^n}{dt^n} \sin^{2j} t. \tag{4.11}$$

Then

$$|^nW_k(t)| \leqq C(n)t^{\{2k-n\}}. \tag{4.12}$$

Proof. Obviously, $^nW_k(t)$ is a trigonometric polynomial. In a neighborhood of $t=0$, we have

$$\sin^{2k} t = t^{2k} + O(t^{2k+2}),$$

and therefore,

$$\left|\frac{d^n \sin^{2k}}{dt^n}(t)\right| \leqq C(n)t^{\{2k-n\}}. \tag{4.13}$$

Hence, for $j \leqq k \leqq n$,

$$\sin^{2(k-j)} t \frac{d^n}{dt^n} \sin^{2j}(t) \leqq C(n)t^{2(k-j)}t^{\{2j-n\}}$$

$$\leqq C(n)t^{A_1(k,j,n)},$$

where

$$A_1(k, j, n) = 2(k-j) + \{2j-n\}. \tag{4.14}$$

Since it is easily checked that

$$A_1(k, j, n) = \{2k-n\},$$

the result follows.

LEMMA 4.4. *We have the following for all* $0<\Delta<\Delta_0$ *and* $]\xi[\geqq \theta$, *where* $\theta = \arcsin \Delta^{1/2}$:

$$\left|\frac{\partial^{|k|} U_\Delta}{\partial \xi_1^{k_1}\partial \xi_2^{k_2}}(\xi)\right| \leqq C(|k|)]\xi[^{-\{|k|-2\gamma\}}. \tag{4.15}$$

Proof. We have (see, for example, [9, p. 19])

$$\frac{\partial^{k_1} U_\Delta}{\partial \xi_1^{k_1}} = \sum_{j=1}^{k_1} {}^{k_1}W_j(\xi_1)\frac{1}{j!}\frac{\partial^j u_\Delta}{\partial x_1^j}(\hat\Phi(\xi)),$$

and therefore

$$\frac{\partial^{|k|} U_\Delta}{\partial \xi_1^{k_1}\partial \xi_2^{k_2}} = \sum_{j=1}^{k_1}\sum_{l=1}^{k_2} {}^{k_1}W_j(\xi_1)^{k_2}W_l(\xi_2)\frac{1}{j!l!}\frac{\partial^{j+l} u_\Delta}{\partial x_1^j\partial x_2^l}(\hat\Phi(\xi)). \tag{4.16}$$

Using Lemma 4.3, we find that

$$\left|\frac{\partial^{|k|} U_\Delta}{\partial \xi_1^{k_1}\partial \xi_2^{k_2}}\right| \leqq C(k_1, k_2)\sum_{j=1}^{k_1}\sum_{l=1}^{k_2} \xi_1^{\{2j-k_1\}}\xi_2^{\{2l-k_2\}}]\hat\Phi(\xi)[^{-\{j+l-\gamma\}}, \tag{4.17}$$

provided that $]\hat\Phi(\xi)[>\Delta$.

Because $C]\xi[^2 \geqq]\hat\Phi(\xi)[\geqq C]\xi[^2$, (4.17) can be written in the form

$$\left|\frac{\partial^{|k|} U_\Delta}{\partial \xi_1^{k_1}\partial \xi_2^{k_2}}\right| \leqq C(k_1, k_2)\sum_{j=1}^{k_1}\sum_{l=1}^{k_2} \xi_1^{\{2j-k_1\}}\xi_2^{\{2l-k_2\}}]\xi[^{-2\{j+l-\gamma\}}, \tag{4.18}$$

provided that $]\xi[\geqq\theta$. Since $c_1\xi_2 \leqq \xi_1 \leqq c_2\xi_2$ if $\xi \in \operatorname{supp} U_\Delta$, we have

$$\left|\frac{\partial^{|k|} U_\Delta}{\partial \xi_1^{k_1}\partial \xi_2^{k_2}}\right| \leqq C(k_1, k_2)\sum_{j=1}^{k_1}\sum_{l=1}^{k_2}]\xi[^{A_2(j,l,k_1,k_2,\gamma)}, \tag{4.19}$$

where

$$A_2(j, l, k_1, k_2, \gamma) = \{2j-k_1\} + \{2l-k_2\} - 2\{j+l-\gamma\}. \tag{4.20}$$

Since a simple computation shows that

$$A_2 \geqq -\{|k|-2\gamma\}, \tag{4.21}$$

(4.15) follows from (4.19) and (4.21).

We will now prove Theorem 4.1.

Proof of Theorem 4.1. Let

$$R_\alpha^\Phi = \hat\Phi^{-1}[R_\alpha].$$

Since $\operatorname{supp} U_\Delta \subset R_\alpha^\Phi$ by assumption, we have

$$\|U_\Delta\|^2_{k,Q(\pi/2)} \leqq \|U_\Delta\|^2_{k,Q(\theta)\cap R_\alpha^\Phi} + \|U_\Delta\|^2_{k,R_\alpha^\Phi - Q(\theta)}. \tag{4.22}$$

We will now estimate the terms in (4.22).

On $Q(\theta)$, (4.16), Lemma 4.3, and property (iii) show that

$$\left|\frac{\partial^{|m|}U_\Delta}{\partial\xi_1^{m_1}\partial\xi_2^{m_2}}\right|^2 \leqq C(m_1, m_2)\sum_{j=1}^{m_1}\sum_{l=1}^{m_2}\xi_1^{2\{2j-m_1\}}\xi_2^{2\{2l-m_2\}}\Delta^{-2\{j+l-\gamma\}}.$$

Since $\theta\leqq C\Delta^{1/2}$, we find that

$$\begin{aligned}\left\|\frac{\partial^{|m|}U_\Delta}{\partial\xi_1^{m_1}\partial\xi_2^{m_2}}\right\|^2_{0,Q(\theta)} &\leqq C(m_1, m_2)\sum_{j=1}^{m_1}\sum_{l=1}^{m_2}\Delta^{-2\{j+l-\gamma\}+1/2[2\{2j-m_1\}+2\{2l-m_2\}]+1}\\ &\leqq C(m_1, m_2)\sum_{j=1}^{m_1}\sum_{l=1}^{m_2}\Delta^{A_2(j,l,m_1,m_2,\gamma)+1}\\ &\leqq C(m_1, m_2)\Delta^{-\{|m|-2\gamma\}+1}.\end{aligned}\tag{4.23}$$

On $R^\Phi_\alpha - Q(\theta)$, Lemma 4.4 shows that

$$\begin{aligned}\left\|\frac{\partial^{|m|}U_\Delta}{\partial\xi_1^{m_1}\partial\xi_2^{m_2}}\right\|^2_{0,R^\Phi_2-Q(\theta)} &\leqq C(m_1, m_2)\int_{C\Delta^{1/2}}^{\pi/2}\xi_1\xi_1^{-2\{|m|-2\gamma\}}\,d\xi_1\\ &\leqq C(m_1, m_2)(\Delta^{1/2})^{-\{2\{|m|-2\gamma\}-2\}}.\end{aligned}\tag{4.24}$$

The result now follows.

THEOREM 4.2. *Suppose that $u_\Delta\in\psi_\gamma$ is continuous on $\bar{\hat{Q}}$, $u_\Delta(0,0)=0$, and $u_\Delta=0$ on the side s_3 of T. Then there exists a $z_p\in\mathcal{P}_p(T)$ such that the following hold for integer $k\geqq 2\gamma+1$:*

$$\text{(i)}\ \|u_\Delta - z_p\|_{1,T}\leqq C(k)p^{-(k-2)}\Delta^{-1/2\{k-2\gamma\}+1/2},\tag{4.25}$$

(ii) $Z_p=0$ *on the side* s_3 *of* T,

$$\begin{aligned}\text{(iii)}\ &\left[\int_{s_i}(u_\Delta-z_p)^2\sigma_i^{-1}\,d\sigma_i+\int_{s_i}\left(\frac{d}{d\sigma_i}(u_\Delta-z_p)\right)^2\sigma_i\,d\sigma_i\right]^{1/2}\\ &\leqq C(k)p^{-(k-2)}\Delta^{-1/2\{k-2\gamma\}+1/2}\quad \textit{for } i=1 \textit{ or } 2,\end{aligned}\tag{4.26}$$

where we have used σ_i to denote the distance to the origin measured along s_i.

Proof. Theorem 4.1 shows that the function U_Δ satisfies (4.10). Therefore, there exists a sequence of trigonometric polynomials of period π all symmetric with respect to the lines $\xi_1=\pm\pi/2$, $\xi_1=0$, $\xi_2=\pm\pi/2$, $\xi_2=0$ and are such that

$$\|U_\Delta - t_p\|_{m,\hat{Q}(\pi/2)}\leqq C(m,k)p^{-(k-m)}\Delta^{-1/2\{k-2\gamma\}+1/2}\quad\text{for } 0\leqq m\leqq k.\tag{4.27}$$

Since $U_\Delta=0$ at the vertices of T^Φ, we can modify each t_p so that $t_p=0$ at these vertices, the symmetries remain and (4.27) holds for all $m>1+\varepsilon$. In addition, using the trace theorem, we have

$$\|t_p - U_\Delta\|_{1,s_i^\Phi}\leqq C(k)p^{-(k-2)}\Delta^{-1/2\{k-2\gamma\}+1/2}\quad\text{for } i=1, 2, \text{ or } 3.\tag{4.28}$$

Now let $t_p(\xi)=z_p(\Phi(\xi))$ where z_p is an algebraic polynomial of degree p. Using Lemma 4.1, we obtain (4.25). Since $t_p=0$ at the vertices of T^Φ, $z_p=0$ at the vertices of T. Furthermore, we have that

$$|U_\Delta - t_p(\sigma_i^\Phi)|<C(\sigma_i^\Phi)^{1/2}\|t_p-U_\Delta\|_{1,s_i^\Phi},$$

where we have denoted the distance to the origin measured along s_i^Φ by σ_i^Φ. Thus, we find that

$$\int_{s_i} (z_p - u)^2 \sigma_i^{-1}\, d\sigma_i \leqq C \int_{s_i^\Phi} (t_p - U_\Delta)^2 (\sigma_i^\Phi)^{-2} \sigma_i^\Phi\, d\sigma_i^\Phi$$

(4.29)

$$\leqq C \int_{s_i^\Phi} (t_p - U_\Delta)^2 (\sigma_i^\Phi)^{-1}\, d\sigma_i^\Phi$$

$$\leqq C \|t_p - U_\Delta\|^2_{1,s_i^\Phi} \quad \text{for } i = 1 \text{ or } 2,$$

$$\int_{s_i} \left(\frac{d}{d\sigma_i}(z_p - u_\Delta)\right)^2 \sigma_i\, d\sigma_i \leqq C \int_{\sigma_i^\Phi} \left(\frac{d}{d\sigma_i^\Phi}(t_p - U_\Delta)\right)^2 \sigma_i^\Phi\, d\sigma_i^\Phi$$

(4.30)

$$\leqq \|t_p - U_\Delta\|^2_{1,s_i^\Phi} \quad \text{for } i = 1 \text{ or } 2,$$

and (4.26) follows.

Since the map $\hat{\Phi}$ is analytic on s_3, we can use Lemma 3.1 and (4.28) to modify z_p, so that $z_p = 0$ on s_3. The proof is now complete.

Remark. We have assumed that the triangle T is situated as shown in Fig. 4.1. It is easy to see that a linear transformation of coordinates does not alter Theorem 4.2. Therefore, Theorem 4.2 is true for any triangle T with a vertex at the origin.

4.3. An example of a ψ_γ-family. Let $\chi(x)$ denote an infinitely differentiable function on R such that $\chi(x) = 0$ for $0 < x < \frac{1}{2}$, $0 \leqq \chi(x) \leqq 1$ for $\frac{1}{2} \leqq x \leqq 1$ and $\chi(x) = 1$ for $1 < x < \infty$. Set $\chi_\Delta(x) = \chi(x/\Delta)$ for any $\Delta > 0$.

Let (r, ϕ) denote polar coordinates in R^2 and let

$$u = \rho(r)\theta(\phi), \tag{4.31}$$

be such that $\theta \in C^\infty([-\pi, \pi])$ and $\rho \in C^\infty((0, \infty)) \cap C([0, \infty))$. Furthermore, suppose that u has support in R_α for some $\alpha < 0$, $u = 0$ on s_3, $\rho(0) = 0$ and that there is a $\gamma > 0$ so that

$$\left|\frac{d^n\rho}{dr^n}\right| \leqq r^{\gamma - n} C(n) \quad \text{for all } n \geqq 0 \text{ and } r > 0. \tag{4.32}$$

Now set $\rho_\Delta = \chi_\Delta(r)\rho(r)$ and $u_\Delta = \rho_\Delta\theta(\phi)$. Then, obviously, $u_\Delta = 0$ on s_3 and u_Δ has (compact) support in $\bar{R}_\alpha$ for all $\Delta > 0$.

We will now show that u_Δ forms a ψ_γ-family of functions. Obviously, parts (i) and (ii) of the definition of the ψ_γ-family are satisfied. Also, (4.32) and the fact that $\chi_\Delta = 0$ for $r \leqq \Delta/2$ show that

$$\left|\frac{\partial^k u_\Delta}{\partial r^k}\right| \leqq C \sum_{j=0}^{k} \left|\frac{\partial^j \chi_\Delta}{\partial r^j}\right| \left|\frac{\partial^{k-j}\rho}{\partial r^{k-j}}\right| \leqq C \sum_{j=0}^{k} \Delta^{-j} \Delta^{\gamma - k + j} \leqq C(k)\Delta^{\gamma - k}, \tag{4.33}$$

$$\left|\frac{\partial^{|k|} u_\Delta}{\partial x_1^{k_1} \partial x_2^{k_2}}\right| \leqq C \sum_{l=0}^{|k|} \frac{\partial^l \chi_\Delta \rho}{\partial r^l} \frac{1}{r^{|k|-l}} \leqq C \sum_{l=0}^{|k|} \sum_{j=0}^{l} \left|\frac{\partial^j \chi_\Delta}{\partial r^j}\right| \left|\frac{\partial^{l-j}\rho}{\partial r^{l-j}}\right| \frac{1}{r^{|k|-l}}$$

(4.34)

$$\leqq C \sum_{l=0}^{|k|} \Delta^{\gamma - l} \Delta^{-|k|+l} \leqq C(|k|)\Delta^{\gamma - |k|},$$

for all $k \geqq 0$ and multi-integers α. Thus, since $\chi_\Delta(r) = 1$ if $r \geqq \Delta$, we obtain property (iii) of the definition.

We will now describe another property of the functions discussed in this section.

LEMMA 4.5. *Suppose that u is given by* (4.31), *where ρ and θ satisfy the conditions discussed above and let $u_\Delta = \chi_\Delta(r)\rho(r)\theta(\phi)$. Then*

$$\|u - u_\Delta\|_{1,\hat{Q}} \leqq C\Delta^\gamma \quad \textit{for all } \Delta > 0, \tag{4.35}$$

where C is independent of Δ.

Proof. Set $v = u - u_\Delta = (1 - \chi_\Delta)\rho(r)\theta(\phi)$. Then (4.32) shows that

$$\left|\frac{\partial v}{\partial x_i}\right| \leqq r^{\gamma-1}C \quad \text{for all } 0 < \Delta \leqq r, \tag{4.36}$$

and $v = 0$ for $r \geqq \Delta$. Therefore, we have

$$\int_{\hat{Q}} \left(\frac{\partial v}{\partial x_i}\right)^2 dx \leqq C \int_0^\Delta r^{2\gamma-2} r\, dr \leqq C\Delta^{2\gamma}, \tag{4.37}$$

$$\int_{\hat{Q}} |v|^2\, dx \leqq C \int r^{2\gamma+1}\, dr \leqq C\Delta^{2\gamma+2}, \tag{4.38}$$

and the result easily follows.

From the point of view of applications, the function $\rho(r) = r^{\gamma_0} g(|\log r|)$ is of importance, especially when, for example, $g(x) = x^p$ or $g(x) = \cos x$. For these examples, (4.32) is satisfied for any $\gamma < \gamma_0$.

4.4. The convergence rate of the p-version of the finite element method. Returning to our model problem (3.1) and (3.2), we can assume (see, for example, [10] or [15]) that its solution u_0 can be written in the form

$$u_0 = \omega + \sum_{i=1}^{n} v_i, \tag{4.39}$$

where $\omega \in H^k(\Omega_0)$, $v_1, v_2, \cdots, v_n \in H^k(\Omega_0)$ if $\Gamma u = \partial u/\partial n$, or where $\omega \in H^k(\Omega_0)_0H_0^1(\Omega)$, $v_1, v_2, \cdots, v_n \in H_0^1(\Omega_0)$ if $\Gamma u = u$. We can also assume that each v_i can be decomposed as follows:

$$v_i = a_i\rho_i(r_i)\theta_i(\phi_i), \tag{4.40}$$

where (r_i, ϕ_i) denotes polar coordinates centered at the vertex of the polygon Ω_0, a_i is a constant and $\theta_i \in C^\infty([-\pi, \pi])$. Each ρ_i can be further decomposed as follows:

$$\rho_i(r) = r^{\gamma_i} g_i(|\log r_i|), \tag{4.41}$$

where $\gamma_i > 0$ and the function g_i satisfies the following for all $j \geqq 0$ and $0 < x < \infty$:

$$\left|\frac{\partial^j g_i(x)}{\partial x^j}\right| \leqq C_i(j)x^{p_{i,j}} + D_i, \quad p_{i,j} \geqq 0.$$

The coefficient γ_i is closely related to the angle of the boundary of Ω at the vertex A_i. Without any loss of generality, we can assume that each θ_i is a smooth periodic function with period 2π. Thus, the function v_i can be extended to all of R^2. The form described in (4.39) occurs in all elliptic problems, for instance, problems of elasticity (see [15]).

By using suitable partitions of unity and suitable modifications of ω, we can assume each ρ_i has small support and that each v_i is associated with an (open) cone K_i (of angle $< \pi$) which contains the support of v_i and only one triangle that meets A_i.

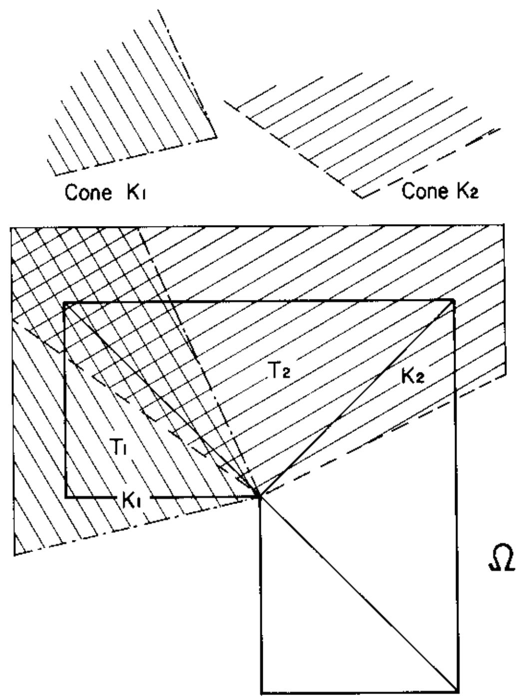

FIG. 4.2. *Triangles and enclosing cones.*

Let $\chi_{i,\Delta}=\chi((r-r_i)/\Delta)$ for $1\leqq i\leqq n$, and let

$$V_\Delta=\sum_{i=1}^{n} v_i\chi_{i,\Delta}=\sum_{i=1}^{n} v_{i,\Delta}.$$

We now state the major theorem of this section.

THEOREM 4.3. *Let u_0 be the exact solution of the problem* (3.1), (3.2) *which, as we have noted, can be written in the form described by* (4.39), (4.40) *and* (4.41). *Let u_p be the finite element approximate for u_0. Then for each $k>1$ and $\varepsilon>0$, we have*

(4.42) $$\|u_0-u_p\|_{1,\Omega_0}\leqq C(\varepsilon)p^{-\mu+\varepsilon}\|f\|_{k,\Omega_0}.$$

(4.43) $$\text{where } \mu=\min\{k+1, 2\gamma_1,\cdots,2\gamma_n\}$$

and where $\alpha_i=\pi/\gamma_i$ is the angle made by $\partial\Omega$ at the vertex A_i, for $1\leqq i\leqq n$.

Proof. The exact solution u_0 can be written in the form described in (4.39) where $\omega\in H^{k+2-\varepsilon}(\Omega_0)$ and $\gamma_i=\pi/\alpha_i$(see [15]). It is sufficient to show that functions ω and $\{v_i\}$ can each be approximated by some $z_p\in\mathcal{P}_{p,0}^{[\mathcal{S}]}$ if $\Gamma u=u$, or by some $z_p\in\mathcal{P}_p^{[\mathcal{S}]}$ if $\Gamma u=\partial u/\partial n$, in such a way so as to preserve (4.42). Theorems 3.2 and 3.1 show that the function ω is approximable in the desired fashion. Thus, we only have to concentrate on the approximation of some function $v_i\in H_0^1(\Omega_0)$ or $H^1(\Omega_0)$. It is easy to see that $v_{i,\Delta}\in H_0^1(\Omega_0)$ if $v_i\in H_0^1(\Omega_0)$ and using Lemma 4.5, we see that

(4.44) $$\|v_{i,\Delta}-v_i\|_{1,\Omega_0}\leqq C\Delta^{\gamma_i-\varepsilon}\quad\text{for any }\Delta>0.$$

In addition, Theorem 4.2 and the remark following it show that if T_j has A_i as a vertex, there exist polynomials

$$\tilde{z}_{p,j}\in\mathcal{P}_p(T_j)\quad\text{for } p\geqq 0,$$

such that for any $\tilde{k}_i \geqq 2\gamma_i + 1$, we have

$$\|\tilde{z}_{p,j} - v_{i,\Delta}\|_{1,T_j} \leqq C(\tilde{k}_i)p^{-(\tilde{k}_i-2)}\Delta^{-1/2\{\tilde{k}_i-2\gamma_i\}+1/2}, \tag{4.45}$$

and such that $\tilde{z}_{p,j}$ satisfies (4.26) on the sides of T_j. If T_{j_1} and T_{j_2} have A_i as a common vertex and a common edge s, the function $\tilde{z}_{p,j_1} - \tilde{z}_{p,j_2}$ is a polynomial of degree p on s which vanishes at the endpoints of s and which satisfies (4.26). We can use Lemma 4.2 to alter $\tilde{z}_{p,j_1}$ so that $\tilde{z}_{p,j_1} = \tilde{z}_{p,j_2}$ on s such that (4.45) still holds. A similar alteration can be made if $v_{i,\Delta} = 0$ on a side $s \subset \partial\Omega_0 \cap \bar{T}_j$ for some T_j having A_i as a vertex. Thus, we can construct a $z_{p,i} \in \mathcal{P}_p^{[\mathcal{S}]}(\Omega_0)$ or a $z_{p,i} \in \mathcal{P}_{p,0}^{[\mathcal{S}]}(\Omega_0)$ if $v_i \in H_0^1(\Omega_0)$ such that

$$\|z_{p_i} - v_{i,\Delta}\|_{1,\Omega_0} \leqq C(\tilde{k}_i)p^{-(\tilde{k}_i-2)}\Delta^{-1/2\{\tilde{k}_i-2\gamma_i\}+1/2}. \tag{4.46}$$

Combining (4.44) and (4.46), we find that if $\tilde{k}_i \geqq 2\gamma_i + 1$, then

$$\|z_{p,i} - v_i\|_{1,\Omega_0} \leqq C(\varepsilon)\Delta^{\gamma_i-\varepsilon} + C(\tilde{k}_i)p^{-(\tilde{k}_i-2)}\Delta^{-1/2\{\tilde{k}_i-2\gamma_i\}+1/2}.$$

Now choose $k_0 \geqq \max_{1\leqq i\leqq n}(2\gamma_i + 1)$ so that

$$\lambda = \frac{-2 + k_0}{(1/2)k_0 - (1/2)} \geqq 2 - \varepsilon,$$

and set $\Delta = p^{-\lambda}$. We thus have the following for each $1 \leqq i \leqq n$:

$$\begin{aligned} \|z_{p,i} - v_i\|_{1,\Omega} &\leqq C(\varepsilon)p^{-2\gamma_i+\varepsilon'} + C(k_0)p^{-k_0+2-\gamma_i\lambda+\lambda(k_0/2-1/2)} \\ &\leqq C(\varepsilon)p^{-2\gamma_i+\varepsilon'} + C(k_0)p^{-2\gamma_i+\varepsilon'} \\ &\leqq C(\varepsilon)p^{-2\gamma_i+\varepsilon'}. \end{aligned} \tag{4.47}$$

The proof is now easily completed. (It has been assumed, for simplicity, that $\|f\|_{k,r_0} = 1$.)

We have only analyzed the rate of convergence of the p-version for the model problem (3.1) and (3.2). However, since $H^1(\Omega_0)$ and $H_0^1(\Omega_0)$ approximation results were proven, this restriction is not an essential one.

Note that (4.42) and (4.43) show that Theorems (3.6) and (4.3) prove the best possible convergence rates, up to an arbitrarily small $\varepsilon > 0$.

5. Numerical examples. In order to illustrate the results of the theorems and in order to show the efficiency of the p-version of the finite element, we now present several examples. The first example is a simple bar problem in one-dimension and the numerical results are based on a computer program written specifically for this problem. The other examples are two-dimensional and the numerical results are based on COMET-X, an experimental prototype for a general purpose finite element computer program developed at Washington University, which implements the p-version of the finite element method [2].

5.1. A one-dimensional (bar) problem. We consider the problem:

$$u'' = -q(x) \quad \text{for } x \in \Omega = (-1, 1), \quad \left(' = \frac{d}{dx}\right), \tag{5.1}$$

where the (loading) function $q(x)$ and the (Dirichlet) boundary conditions will be specified later. The energy inner product is

$$B(u, v) = (u, v)_E = \int_{-1}^{1} u'(x)v'(x)\,dx. \tag{5.2}$$

We seek a solution $u \in H_0^1(\Omega)$ which satisfies

$$(5.3) \qquad (u, v)_E = \int_{-1}^{1} u'(x)v'(x)\,dx = \int_{-1}^{1} q(x)v(x)\,dx \quad \text{for all } v \in H_0^1(\Omega).$$

We choose as basis functions

$$\psi_i(x) = \int_{-1}^{x} P_i(t)\,dt \quad \text{for } i \geqq 1,$$

where $P_i(t)$ is the Legendre polynomial of degree i. Observe that $\psi_i(x)$, $i = 1, 2, \cdots$ form an orthogonal family with respect to the energy inner product; i.e., $(\psi_i, \psi_j)_E = \int_{-1}^{1} P_i(x)P_j(x)\,dx = (2/(2i+1))\delta_{ij}$.

In this one-dimensional case, it is possible to prove direct and inverse approximation theorems by using weighted Sobolev (respectively Besov) spaces associated with the Legendre differential equation,

$$-\frac{d}{dx}\left[(1-x^2)\frac{du}{dx}\right] = n(n+1)u,$$

once we realize that the Legendre polynomials are eigenfunctions of this equation. Using this approach, ε does not appear in the expressions for the rate of convergence, e.g., in (3.35) and (4.42). It is not clear how to generalize this idea of the two-dimensional case. Our proof for two dimensions was therefore quite different.

First we consider convergence when Ω is not divided; i.e., we use only one interval. The finite element solution $u_p \in \mathscr{P}_{p,0}^{[\mathscr{S}]}(\Omega)$ satisfies

$$(5.4) \qquad (u_p, \psi_i)_E = \int_{-1}^{1} q(x)\psi_i(x)\,dx, \qquad i = 1, 2, \cdots, p.$$

If we write

$$u_p(x) = \frac{1-x}{2}u(-1) + \frac{1+x}{2}u(1) + \sum_{i=1}^{p} a_i\psi_i(x),$$

it follows that

$$(5.5) \qquad a_i = \frac{2i+1}{2}\int_{-1}^{1} q(x)\psi_i(x)\,dx, \quad i = 1, 2, \cdots, p.$$

Also, if we denote the error by

$$(5.6) \qquad e_p(x) = u(x) - u_p(x),$$

it follows that in the energy norm $\|e_p\|_E^2 = (e_p, e_p)_E$,

$$(5.7) \qquad \begin{aligned} \|e_p\|_E^2 = \|u-u_p\|_E^2 = \|u\|_E^2 - \|u_p\|_E^2 &= \left\| \sum_{i=p+1} a_i\psi_i(x) \right\|_E^2 \\ &= \sum_{i=p+1} a_i^2 \frac{2}{2i+1}. \end{aligned}$$

If we let $U = \|u\|_E^2$ denote the strain energy, then $U - U_p = \|e_p\|_E^2$ is the error in strain energy.

Case A. $du/dx = \sqrt{1-x^2}$, $q(x) = -(d/dx)\sqrt{1-x^2}$.

In this case, $u(x)=\frac{1}{2}(x\sqrt{1-x^2}+\sin^{-1} x)$ and the boundary conditions are $u(-1)=-\pi/4$, $u(1)=\pi/4$. Also, the energy is

$$\|u\|_E^2=\int_{-1}^{1}(1-x^2)\,dx=\frac{4}{3}.$$

The coefficients a_i in (5.5) can be evaluated explicitly. First, (5.5) becomes

$$a_i=\frac{2i+1}{2}\int_{-1}^{1}\sqrt{1-x^2}P_i(x)\,dx. \tag{5.8}$$

Now, $a_i=0$ for i odd, and using the recurrence relation for derivatives of Legendre polynomials [1],

$$P'_{i+1}(x)-P'_{i+1}(x)=(2i+1)P_i(x),$$

for $i=2m$, $m=1,2,\cdots$, we obtain

$$\begin{aligned}\int_{-1}^{1}\sqrt{1-x^2}P_{2m}(x)\,dx&=-\frac{1}{4m+1}\int_{-1}^{1}\frac{x}{\sqrt{1-x^2}}(P_{2m+1}(x)-P_{2m-1}(x))\,dx\\&=\frac{1}{4m+1}\int_0^{\pi}\cos\theta(P_{2m+1}(\cos\theta)-P_{2m-1}(\cos\theta))\,d\theta.\end{aligned} \tag{5.9}$$

From [1, formula (22.13.7), p. 785], we have

$$\int_0^{\pi}(\cos\theta)P_{2m+1}(\cos\theta)\,d\theta=\frac{\pi}{4^{2m+1}}\binom{2m}{m}\binom{2m+2}{m+1}. \tag{5.10}$$

Substituting (5.9) and (5.10) into (5.8), we obtain by straightforward calculation,

$$a_{2m}=\frac{2(2m)+1}{2}\,\frac{\pi}{4^{2m}2(m+1)(2m-1)}\binom{2m}{m}^2.$$

From Stirling's formula, it follows that

$$\binom{2m}{m}\sim\frac{1}{\sqrt{\pi m}}4^m,$$

so that

$$a_{2m}=O\left(\frac{1}{m^2}\right)\quad\text{as } m\to\infty. \tag{5.11}$$

Therefore, the square of the energy of the error in (5.7) is given by

$$\|e_p\|_E^2=\sum_{i=p+1}a_i^2\frac{2}{2i+1}=O\left(\sum_{i=p+1}\frac{1}{i^5}\right)=O\left(\frac{1}{p^4}\right)=O\left(\frac{1}{N^4}\right), \tag{5.12}$$

where N denotes the number of degrees of freedom ($p\approx N$ in one dimension).

On the other hand, in order to study the convergence of the (usual) h-version with N linear uniformly distributed elements, let $x_i=-1+(2/N)i$, $i=0,1,2,\cdots,N$ and let $u_h(x)$ denote the corresponding finite element solution. Then

$$u_h(x_i)=u(x_i),\qquad i=0,1,2,\cdots,N,$$

and we can compute the norm of the error $e_h=u(x)-u_h(x)$. We obtain (for linear elements)

$$O\left(\frac{1}{N^2}\right)|\log N| \leqq \|e_h\|_E^2 = O\left(\frac{1}{N^2}\right)|\log N|. \tag{5.13}$$

For quadratic and higher elements, we obtain

$$O\left(\frac{1}{N^2}\right) \leqq \|e_h\|_E^2 = O\left(\frac{1}{N^2}\right) \tag{5.14}$$

(see [3]).

Fig. 5.1 shows on a log scale the behavior of the square of the energy error. We see that in the case of the p-version, the rate is practically 4, as follows from the asymptotic

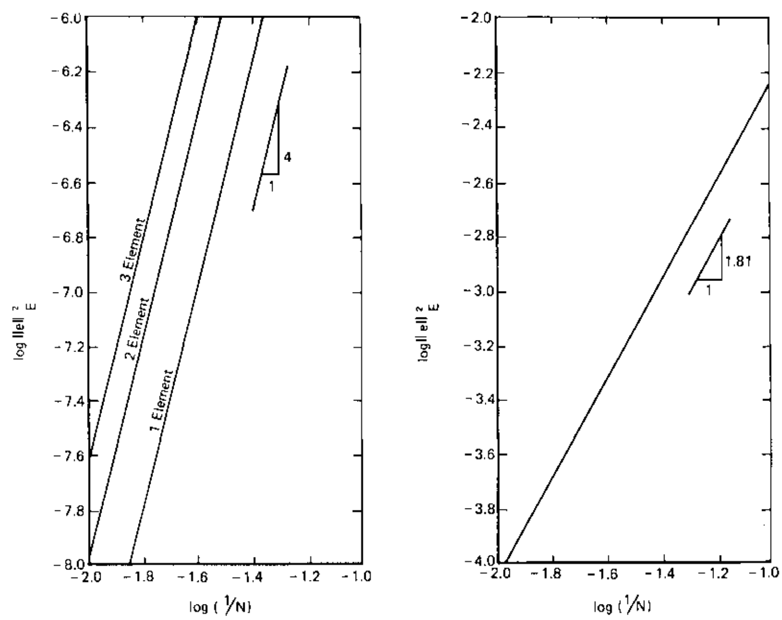

FIG. 5.1. *Case* A: *Square of the energy error vs. reciprocal of the number of degrees of freedom.*

analysis. In the case of the h-version, the asymptotic range is not achieved and we see the rate is about 1.81 instead of 2.

Case B. $u(x) = |x|^{3/2}(1-x^2)$, $q(x) = -(d^2/dx^2)(|x|^{3/2}(1-x^2))$.

The boundary conditions are $u(-1) = u(1) = 0$. The only qualitative difference between this case and Case A is that the square root singularity in $u'(x)$ now occurs in the interior of Ω instead of at its boundary.

We again consider one interval using the same basis functions as before. Equation (5.5) now becomes

$$\begin{aligned} a_i &= \frac{2i+1}{2}\int_{-1}^{1} \frac{d}{dx}(|x|^{3/2}(1-x^2))P_i(x)\,dx \\ &= \frac{2i+1}{2}\int_{-1}^{1} |x|^{1/2}(\tfrac{3}{2}-\tfrac{7}{2}x^2)(\operatorname{sgn} x)P_i(x)\,dx \\ &= \begin{cases} 0 & \text{if } i \text{ is even,} \\ \dfrac{2i+1}{2}\displaystyle\int_{0}^{1} x^{1/2}(3-7x^2)P_i(x)\,dx & \text{if } i \text{ is odd.} \end{cases} \end{aligned} \tag{5.15}$$

From [1, formula (22.13.9), p. 786], we have

$$\int_0^1 xP_{2m+1}(x)\,dx=\frac{(-1)^m\Gamma(m+\frac{1}{2}-\lambda/2)\Gamma(1+\lambda/2)}{2\Gamma(m+2+\lambda/2)\Gamma(\frac{1}{2}-\lambda/2)},\qquad \lambda>-2,$$

so that after a straightforward calculation, we obtain

$$\int_0^1 x^{1/2}(3-7x^2)P_{2m+1}(x)\,dx=(-1)^m\left[\frac{\frac{3}{2}\Gamma(m+\frac{1}{4})\Gamma(\frac{5}{4})}{\Gamma(m+\frac{5}{4})\Gamma(\frac{1}{4})}-\frac{\frac{7}{2}\Gamma(m-\frac{3}{4})\Gamma(\frac{9}{4})}{\Gamma(m+\frac{13}{4})\Gamma(-\frac{3}{4})}\right]$$

$$=(-1)^{m+1}\left[\frac{\Gamma(m-\frac{3}{4})\Gamma(\frac{5}{4})(2m+1)(m+1)}{\Gamma(m+\frac{13}{4})\Gamma(-\frac{3}{4})}\right].$$

After substitution in (5.15), we find that

$$a_{2m+1}=(-1)^{m+1}\frac{\Gamma(m-\frac{3}{4})\Gamma(\frac{5}{4})(2m+1)^2(m+1)}{\Gamma(m+\frac{13}{4})\Gamma(-\frac{3}{4})}.$$

From Stirling's formula, it follows that for i odd,

$$a_i=O\left(\frac{1}{i}\right)\quad \text{as } i\to\infty.$$

Therefore, the square of the energy of the error is now given by

$$\|e_p\|_E^2=\sum_{i=p+1}a_i^2\frac{2}{2i+1}=O\left(\sum_{i=p+1}\frac{1}{i^3}\right)=O\left(\frac{1}{p^2}\right)=O\left(\frac{1}{N^2}\right).$$

Thus, we obtain the same rate of convergence (up to a log term) for the square of the error $\|e_h\|_E^2$ as obtained for the h-version. This illustrates the importance of the statement made at the end of § 4, that in order to get the full power of the p-version, singularities must be located at vertices of the finite element mesh.

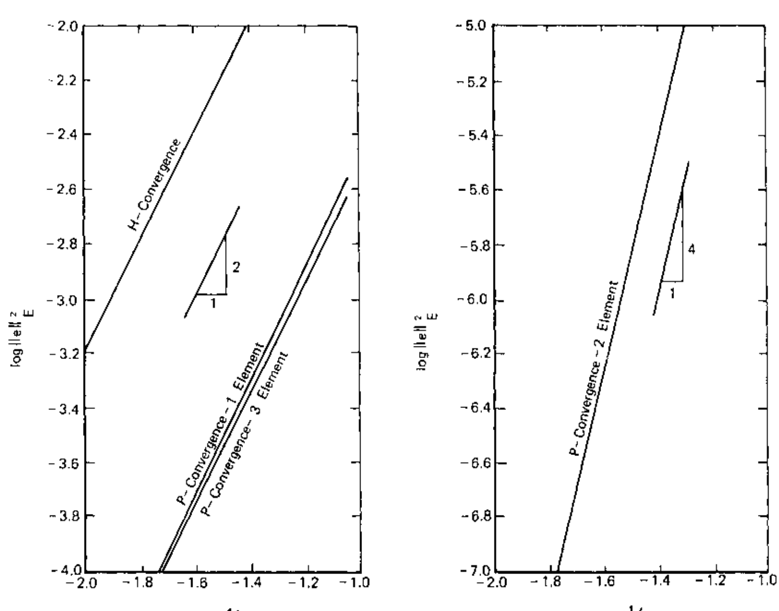

FIG. 5.2. *Case B: Effect of the location of singularity on the rate of convergence.*

To illustrate this point further, we plot in Figs. 5.2, $\|e_h\|_E^2$ and $\|e_p\|_E^2$ for Case B, using one, two and three equal intervals for the p-version of the finite element method. The results are summarized in Table 5.1. The convergence of the h-version remains the same ($\|e_h\|_E^2 = O((1/N^2)\log N)$ for both Cases A and B). In Case A, the convergence of

TABLE 5.1
Rates of convergence for the h-version (linear elements) and the p-version of the finite element method in a bar problem.

	h-version	p-version					
		One interval		Two intervals		Three intervals	
	$\|e_h\|_E^2$	a_N	$\|e_p\|_E^2$	a_N	$\|e_p\|_E^2$	a_N	$\|e_p\|_E^2$
Case A	$\frac{1}{N^2}\|\log N\|$	$\frac{1}{N^2}$	$\frac{1}{N^4}$	$\frac{1}{N^2}$	$\frac{1}{N^4}$	$\frac{1}{N^2}$	$\frac{1}{N^4}$
Case B	$\frac{1}{N^2}\|\log N\|$	$\frac{1}{N}$	$\frac{1}{N^2}$	$\frac{1}{N^2}$	$\frac{1}{N^4}$	$\frac{1}{N}$	$\frac{1}{N^2}$

the p-version remains the same regardless of the number of intervals ($\|e\|_E = O(1/N^2)$), whereas in Case B, the order is 2 for *two* intervals, whereas it is only 1 for both *one* and *three* intervals. This is, of course, because in Case B for two intervals the singularity is at a vertex of the mesh, whereas for one and three intervals, it is in the interior of elements of the mesh, with the consequent degrading of the rate of convergence. In Case A, the singularity is always at a vertex of a mesh. We mentioned here only the case of the h-version with *uniform* mesh spacing. It can be shown that when an *optimal nonuniform* mesh spacing for elements of degree p(fixed) is used, then $\|e_h\|_E^2 = O(1/N^p)$, where the order relation depends on p. In this very special case, it is possible, of course, to analyse in more detail the combined h-p-version, but we shall not go into that.

5.2. Two-dimensional problems. An edge-cracked panel and a parabolically-loaded panel. We now consider two problems taken from two-dimensional linear elasticity. One is an edge-cracked rectangular panel, shown in Fig. 5.3; the other is the parabolically-loaded square panel, shown in Fig. 5.4. In both cases, the displacement

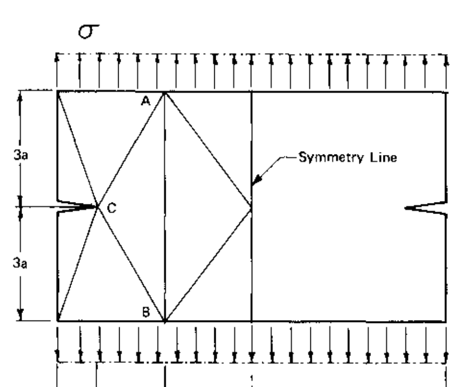

FIG. 5.3. *Edge-cracked rectangular panel and finite element triangulation.*

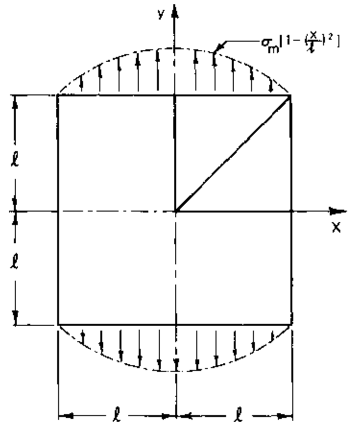

FIG. 5.4. *Parabolically-loaded square panel and finite element triangulation.*

field is locally of the form $\underset{\sim}{u} = r^\alpha \phi(\theta)$, where r and θ denote suitable polar coordinates and ϕ is a smooth function. In the case of the edge-cracked panel, $\alpha = \frac{1}{2}$ when r is measured from the crack tip; in the case of the parabolically-loaded panel, $\alpha \cong 2.74$ when r is measured from a corner of the panel (see [27]). The computations were performed with the computer program COMET-X which allows the polynomial order p to be varied between 1 and 8. We wish to illustrate the following points:

(a) As claimed by the theoretical results, the rate of convergence is

$$U - U_p \sim CN^{-2\alpha}, \tag{5.16}$$

(when neglecting ε and the fact that the edge-cracked panel is not a Lipschitzian domain). In Fig. 5.5, we plotted $\tilde{U} - U_p$ vs. N^{-1} on a log–log scale for the edge-cracked

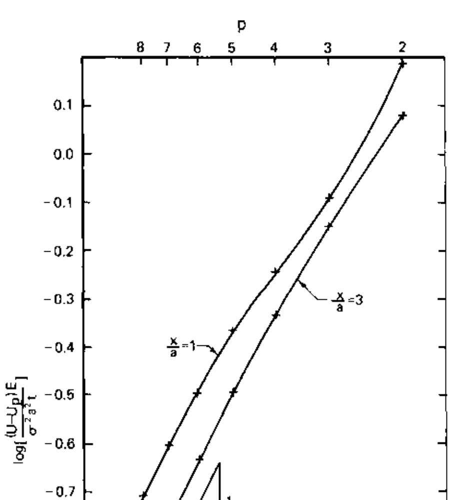

FIG. 5.5. *Edge-cracked rectangular panel. Estimated error in strain vs. reciprocal of the number of degrees of freedom.*

panel for two x/a ratios. $\tilde{U}$ is an estimate of the exact strain energy value (of the half-panel) obtained by extrapolation from the following expression:

$$\tilde{U} = \max_{0<x/a<4} \frac{U_8N_8 - U_7N_7}{N_8 - N_7} = 17.385486\frac{\sigma^2 a^2 t}{E}, \tag{5.17}$$

in which the subscripts indicate polynomial orders, σ is the applied stress, t is the panel thickness, E is the modulus of elasticity. Poisson's ratio was 0.3 in all computations. It is seen that the slopes of the log $(\tilde{U} - U_p)$ curves rapidly approach $2\alpha = 1$. Significantly, the asymptotic range is entered at low, computable p values. This has been utilized in practical computations [26]. Similar behavior is observed for the parabolically-loaded square panel in Fig. 5.6. Here the slope of log $(U - U_p)$ approaches $2\alpha = 5.48$. For this problem, a series solution is available and U can be computed to arbitrary precision [8].

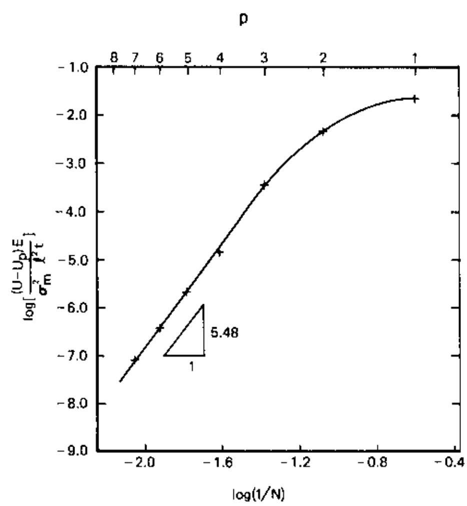

FIG. 5.6. *Parabolically-loaded square panel. Error in strain energy vs. reciprocal of the number of degrees of freedom.*

(b) When the singularity is not located at a vertex, the rate of convergence decreases. To illustrate this feature, we varied the parameter x for the edge-cracked panel (Fig. 5.3) and computed 2α in (5.16) from the 7th- and 8th-order approximations:

$$2\alpha_{7-8} = \frac{\log \dfrac{\tilde{U} - U_8}{\tilde{U} - U_7}}{\log \dfrac{N_7}{N_8}},$$

for various x/a ratios. The results are plotted in Fig. 5.7. It is seen that $2\alpha_{7-8}$ decreases as the interelement boundary approaches the crack tip C. It was found that aspect ratios as high as 300 could be employed without encountering numerical instability.

5.3. Roundoff error. When high-order polynomials are used, the choice of basis functions becomes important from the point of view of roundoff error. It is possible to design stable basis functions on the basis of theory developed mainly by Mikhlin (see [16, Chapt. 2] and [4, Chapt. 4, 7]). Of course, the choice of basis functions is also influenced by programming considerations and the range of p for which the program is

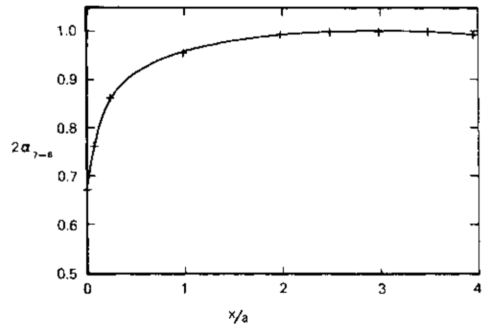

FIG. 5.7. *Edge-cracked panel. Variation of the rate of convergence with x/a.*

written. In general, it is desirable that the basis functions be hierarchic, as described in § 6.1, and the computation of elemental stiffness matrices and load vectors be as simple as possible.

The basis functions currently in COMET-X were chosen primarily on the basis of programming considerations and they are not optimal from the point of view of roundoff error. Experience with the code has not indicated significant accumulation of roundoff error in double precision computations within the range of p allowed by COMET-X (1 to 8), however.

To study the characteristics of these basis functions from the point of view of roundoff error, the assembly and elimination procedures were executed in both double and single precision (7 versus 15 decimals on the DEC system 20 computer) for the two problems described in § 5.2. All other computations were performed in double precision only. (COMET-X employs a modified version of Irons' frontal solver [11] to carry out assembly and elimination.) The results, given in Table 5.2 indicate that for $p \leqq 8$ the roundoff error is not critical, but if significantly higher p is to be used, then it will be necessary to exercise caution in selecting the basis functions.

TABLE 5.2.
Accumulation of roundoff error with increasing polynomial order.

p	Edge-cracked rectangular panel (Fig. 5.3) $x/a=3$			Edge-cracked rectangular panel (Fig. 5.3) $x/a=0.2$			Parabolically-loaded square panel (Fig. 5.4)		
	U_{DP}	U_{SP}	$\log \varepsilon$	U_{DP}	U_{SP}	$\log \varepsilon$	U_{DP}	U_{SP}	$\log \varepsilon$
5	17.16690	17.16690	<-6	16.83520	16.83520	<-6	0.2542124	0.2542124	<-6
6	17.25347	17.25329	−4.99	16.97998	16.98000	−5.94	0.2542144	0.2542144	<-6
7	17.31114	17.31670	−3.50	17.04281	17.07217	−2.78	0.2542147	0.2542147	<-6
8	17.34988	17.38613	−2.68	17.12741	17.08960	−2.67	0.2542148	0.2542170	−5.06

U_{DP}: computed strain energy, all computations in double precision;
U_{SP}: computed strain energy, assembly and elimination in single precision;

$$\varepsilon = \frac{|U_{DP} - U_{SP}|}{U}$$

U: exact (or estimated true) strain energy; $U = 17.485486\ \sigma^2 a^2 t/E$ for the edge-cracked panel, $U = 0.25421481\ (\sigma_m^2 l^2 t)/E$ for the parabolically-loaded square panel.

6. Computer implementation of the p-version: COMET-X. In order to implement the p-version efficiently, it is necessary to have available a family of finite elements of arbitrary polynomial degree having certain properties. The family should allow, for example, as much information as possible to be carried over when increasing the degree of the polynomials p to $p+1$. The present version of COMET-X contains a family of triangular finite elements which enforce C^0 continuity across interelement boundaries for problems which require solutions in $\mathring{H}^1(\Omega)$ (planar elasticity). We now describe some of the salient features of COMET-X.

6.1. Hierarchic property of basis functions. The basis functions corresponding to approximation by polynomials of degree p constitute a subset of those corresponding to an approximation by polynomials of degree $p+1$. Therefore, the stiffness matrix for elements of degree p is embedded in the stiffness matrix for elements of degree $p+1$. All calculations performed in generating the pth order elemental stiffness matrices and load vectors can be saved for use in the $(p+1)$st degree calculation. We call this the hierarchic property of the family.

As an illustration of the difference between conventional and hierarchic basis functions, we will consider linear and quadratic C^0 basis functions for a triangle (given in natural coordinates (L_1, L_2, L_3); see [20] for a discussion of natural coordinates). The linear function which is one at vertex i and zero at the other two vertices is L_i for $i = 1, 2$, or 3, and it is the basic function for the nodal variable $u(i)$, $i = 1, 2, 3$. In defining quadratic approximations, conventional approaches use the nodal variables $u(i)$, and $u(i')$ where $i, i' = 1, 2, 3$, where i' is the midpoint of side i (opposite vertex i). It is clear that the basis functions corresponding to $u(i)$, $i = 1, 2, 3$, change from the linear to the quadratic approximation. In the hierarchic approach, the nodal variables used for the quadratic approximation are $u(i)$ and $u_{ss}(i')$, where the subscript s denotes differentiation in the direction of a side. For $p \geqq 3$, the external nodal variables used to enforce C^0 continuity are jth order derivatives at the midpoint of each side in the direction of the side, for $3 \leqq j \leqq p$. Other nodal variables (called internal nodal variables) are used to complete the polynomial to one of degree p (see [12], [13], [14], [19], [20], [21], [22]).

6.2. Precomputed arrays. It is possible to compute certain elemental stiffness submatrices (corresponding to a standard triangle [14]) once and for all, and then to use these standard submatrices in order to calculate the element stiffness matrices in a given problem. Precomputed arrays based on hierarchic families permit convenient use of elements of different polynomial degrees in the same mesh because two elements of different degrees are easily matched along an interelement boundary. The precomputed standard submatrices are also hierarchic in character so that one version of these arrays, corresponding to the maximum polynomial degree which will be used, can be easily stored on a permanent file. Precomputed arrays are described in [23] and have been incorporated into COMET-X.

6.3. Computational cost. There are three main phases in the computational process of the finite element method:

a) Input phase, which includes the computations of elemental stiffness matrices and load vectors;

b) Solution phase, which comprises the assembly and elimination processes;

c) Output phase, which includes the computation of displacements, stresses, etc.

When the number of degrees of freedom is progressively increased, the major variable cost occurs in the solution phase. In a number of numerical experiments performed with COMET-X, it was found that the CPU time for the solution phase can be closely approximated by an expression of the form $a+bN^{\beta}$, where $2<\beta<2.4$ and a and b are constants. Thus, although the stiffness matrix tends to be more fully populated in the p-version than in the h-version, sparse matrix solution techniques have provided a substantial reduction in the number of operations as compared with solvers which do not account for sparsity ($\beta=3$). As has been already noted, the solution technique in COMET-X is similar to Irons' frontal solver technique.

Solution time information is given in Fig. 6.1 for the edge-cracked rectangular panel ($x/a=3$). The computations were performed in double precision on a DEC–20

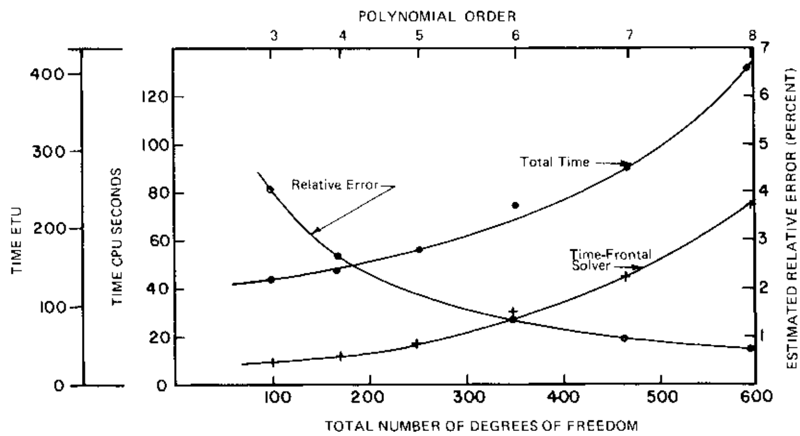

FIG. 6.1. *Edge-cracked panel, $x/a=3$. Variation of the estimated relative error in strain energy and solution time with the number of degrees of freedom. The relative error is defined as $((\tilde{U}-U_p)/\tilde{U})\times 100$ where $\tilde{U}$ is given by* (5.17) *and U_p is the computed strain energy.*

computer, (DEC System 2040, 128K 36-bit word memory, TOPS–20 operating system). The time for the frontal solver includes both the assembly and elimination procedures. The time is given in both CPU seconds and in Equivalent Time Units (ETU). As in [23], an ETU is the time required for squaring a full 18×18 matrix by means of the subroutine GMPRD (double precision) of the IBM Scientific Subroutine Package. On the Dec–20 computer, this operation requires approximately 0.33 seconds.

The total time accounts for all three phases of the computation, including computation of the displacement vector and stress tensor at six points per element.

6.4. The *h*- and *p*-versions of the finite element method. Let us compare the h- and p-versions of the finite element method on the basis of the present state of theory and experience.

1) Asymptotic rate of convergence (in energy) with respect to the number of degrees of freedom:

a) *For smooth solutions*, the rate of convergence of the p-version is not limited by a fixed polynomial degree, as in the h-version.

b) In the case of *nonsmooth solutions*, the p-version has at least the same rate of convergence as the h-version (when the h-version is based on quasi-uniform mesh refinement), but in practical cases, for example, when the singularity is caused by corners, the rate of convergence of the p-version is twice that of the h-version.

c) The h-version, coupled with optimal mesh design, results in higher convergence rate; however, the p-version can also be used in conjunction with optimally designed meshes. In this regard, the mesh design seems to be much less critical for the p-version than for the h-version.

2) Input: Because relatively few elements are used in the p-version, the volume of input data is smaller for the p-version than for the h-version.

3) Roundoff: In practical cases, the roundoff problem does not appear to be more critical for the p-version than for the h-version.

4) Flexibility: From a practical, rather than a theoretical point of view, the flexibility of the p-version is somewhat restricted by the fact that constant coefficients are assumed over large finite element domains. At the present, there is insufficient experience with curvilinear and other numerically integrated elements in connection with the p-version.

5) Solution time: The available experience indicates that for a given number of degrees of freedom, the solution time for the p-version is about the same as for the h-version.

6) Adaptivity: Development of adaptive finite element procedures has now been recognized as an important area for research. (See, for example, [18].) From the point of view of implementation, adaptivity based on the p-version appears to be simpler. Adaptivity based on the h-version poses difficult data management problems. (See, for example [5], [28].) In principle, it is possible to base adaptivity on a combination of the h- and p-versions but such an approach would again pose difficult data management problems. A more promising approach is to employ mesh grading on a prior basis, either manually or with standard mesh generators, and then to make adaptive changes by adjusting p.

7. Acknowledgment. We wish to thank Mr. David A. Dunavant of the Department of Civil Engineering at Washington University for the computations leading to Figs. 5.1 and 5.2.

REFERENCES

[1] M. ABRAMOWITZ AND L. A. STEGUN, *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series 55, Sixth Printing, 1967.

[2] P. K. BASU, M. P. ROSSOW AND B. A. SZABO, *Theoretical Manual and User's Guide for* COMET-X, Center for Computational Mechanics, Washington University, St. Louis, MO 63130, 1977.

[3] I. BABUSKA, R. B. KELLOG AND J. PITKARANTA, *Direct and inverse error estimates for finite elements with mesh refinement*, Numer. Math., 33 (1979), pp. 447–471.

[4] I. BABUSKA, M. PRAGER AND E. VITASEK, *Numerical Processes in Differential Equations*, Wiley-Interscience, New York, 1966.

[5] I. BABUSKA AND W. C. RHEINBOLDT, *On a system for adaptive parallel finite element computations*, Proceedings of the 1978 Annual Conference of the Association for Computing Machinery, Vol. 1, Washington, DC, December 4–6, 1978, pp. 480–489.

[6] R. BELLMAN, *A note on an inequality of E. Schmidt*, Bull. Amer. Math. Soc., 50 (1944), pp. 734–737.

[7] I. BERGH and J. LOFTSTROM, *Interpolation Spaces: An Introduction*, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[8] G. R. COWPER, Personal communication.

[9] I. S. GRADSHTEYN and I. W. RYZHIK, *Tables of Integrals, Series and Products*, (English translation), Academic Press, New York, London, 1965.

[10] P. GRISVARD, *Behavior of the solution of an elliptic boundary value problem in a polygon or polyhedral domain*, in Numerical Solution of Partial Differential Equations III, B. Hubbard, ed., Academic Press, New York, 1976, pp. 207–274.

[11] B. M. IRONS, *A frontal solution program for finite element analysis*, Int. J. Numer. Methods Engrg., 2(1970), pp. 5–32.

[12] I. N. KATZ, *Integration of triangular finite elements containing corrective rational functions*, Int. J. Numer. Methods Engrg., 11 (1977), pp. 107–114.

[13] I. N. KATZ, A. G. PEANO, M. P. ROSSOW AND B. A. SZABO, *The constraint method for finite element stress analysis*, Proceedings of the World Congress on Finite Element Methods in Structural Mechanics, Bournemouth, England, Oct. 12–17, 1975, pp. 7.1–7.26.

[14] I. N. KATZ, A. G. PEANO AND M. P. ROSSOW, *Nodal variables for complete conforming finite elements of arbitrary polynomial order*, Comput. Math. Appl., 4(1978), pp. 85–112.

[15] V. A. KONDRATEV, *Boundary value problems for elliptic equations with conical or angular points*, Trans. Moscow Math. Soc., 16(1967), pp. 227–313.

[16] S. G. MIKHLIN, *The Numerical Performance of Variational Methods*, Wolters-Noordhoff Publishing, Groningen, the Netherlands, 1971.

[17] S. M. NIKOL'SKII, *Approximation of Functions of Several Variables and Imbedding Theorems*, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[18] J. T. ODEN AND K. J. BATHE, *A commentary on computational mechanics*, Appl. Mech. Rev., 31(1978), pp. 1053–1058.

[19] A. G. PEANO, *Hierarchies of Conforming Finite Elements*, DSc. thesis, Department of Civil Engineering, Washington University, St. Louis, MO, 1975.

[20] ———, *Hierarchies of conforming finite elements for plane elasticity and plate bending*, Comput. Math. Appl., 2(1976), pp. 211–224.

[21] A. G. PEANO, I. N. KATZ AND B. A. SZABO, *Constraint Enforcement for Conforming Finite Elements of Arbitrary Polynomial Order*, Report No. DOT–05–03108–4, Washington University, St. Louis, MO 63130, 1975.

[22] M. P. ROSSOW, K. C. CHEN AND J. C. LEE, *Computer implementation of the constraint method*, Comput. & Structures, 6(1976), pp. 203–209.

[23] M. P. ROSSOW AND I. N. KATZ, *Hierarchal finite elements and precomputed arrays*, Int. J. Numer. Methods Engng., 12(1978), pp. 997–999.

[24] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.

[25] B. A. SZABO, *Some recent developments in finite element analysis*, Comput. Math. Appl., 5(1979), pp. 99–115.

[26] B. A. SZABO AND A. K. MEHTA, *P-convergent finite element approximations in fracture mechanics*, Int. J. Numer. Methods Engng., 12(1978), pp. 551–560.

[27] M. L. WILLIAMS, *Stress singularities resulting from various boundary conditions in angular corners of plates in extension*, J. Appl. Mech., 19(1952), pp. 526–528.

[28] P. ZAVE AND W. C. RHEINBOLDT, *Design of an adaptive finite element system*, ACM Trans. Math. Software, 5(1979), pp. 1–17.

[29] J. L. ZOLESIO, *Interpolation d'espaces de Sobolev avec conditions aux limites de type mêlé*, CRAS Paris, 285, October 17, 1977, pp. 621–624.