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THE PLATE PARADOX FOR HARD AND SOFT SIMPLE SUPPORT*

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Abstract. This paper studies the plate-bending problem with hard and soft simple support. It shows that in the case of hard support, the plate paradox, which is known to occur in the Kirchhoff model, is also present in the three-dimensional model and the Reissner-Mindlin model. The paradox consists of the fact that, on a sequence of convex polygonal domains converging to a circle, the solutions of the corresponding plate-bending problems with a fixed uniform load do not converge to the solution of the limit problem. The paper also shows that the paradox is not present when soft simple support is assumed. Some practical aspects are briefly discussed.

Key words. plates, Kirchhoff model, Reissner-Mindlin model, simply supported plate, plate paradox

AMS(MOS) subject classifications. 35J50, 35J55, 73C20, 73C99

1. Introduction. The Kirchhoff model of a plate is usually accepted as a good approximation to the three-dimensional model for thin plates. In the case of simply supported polygonal plates, however, the Kirchhoff model is known to suffer from unphysical phenomena that can lead to a large error of the model in some situations. In particular, the following paradox, referred to below as the plate paradox, occurs [2], [4]. Consider a sequence $\{\omega_n\}$ of convex polygonal domains approaching a circle. For each n, let w_n be the transverse deflection corresponding to the Kirchhoff model of the plate-bending problem, where the plate occupying the region ω_n is simply supported on $\partial\omega_n$ and is subject to a uniform load $p(x)\equiv 1$. Furthermore, let w_c be the solution to the limit problem, i.e., that on the circle. Then as $n\to\infty$, the sequence $\{w_n\}$ converges pointwise, but the limit w_∞ is different from w_c. For example, at the center of the circle the error of w_∞ is about 40 percent. Some other related plate paradoxes are given in [14], [15]. Practical implications occur, for example, in the finite-element method when the domain is approximated by a polygon with sidelength $h\to 0$. For further aspects see also [8], [18], [21], [23], and [25].

It is often assumed that the plate paradox is caused by the assumption of vanishing vertical shear strains that is implicit in the Kirchhoff model. This has been supported, e.g., by a note (see [3]) that the paradox is not present when the Reissner-Mindlin model instead of the Kirchhoff model is used. The aim of this paper is to locate the source of the paradox more precisely. We show that it is the way the *boundary conditions* are imposed in the Kirchhoff model that causes the paradox, and not the overall assumption of vanishing shear strains.

In the three-dimensional model of the plate, the boundary condition of simple support is typically imposed by requiring that the vertical component of the displacement (or at least its average in the vertical direction) vanish on the edge of the plate. On the other hand, the Kirchhoff model effectively imposes the more restrictive condition that all tangential displacements must vanish on the edge. Of course, it is

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also possible to impose such "hard" boundary conditions in other plate models, e.g., in the Reissner–Mindlin model (cf. [22]) or in the three-dimensional model itself. We show that in such a case, *the plate paradox occurs* in both the Reissner–Mindlin model and in the three-dimensional model. On the other hand, we also show that *the paradox does not occur* in these models in case of "soft" support, where only the vertical displacements are restricted on the edge of the plate. Hence, we are led to the conclusion that the paradox is caused by the hard boundary conditions that are intrinsic in the Kirchhoff model.

Our results are based on energy estimates relating the three-dimensional model and the Reissner–Mindlin model to the Kirchhoff model. Such estimates can be derived by combining the energy and complementary energy principles associated to the plate-bending problem. They were, in fact, applied early by Morgenstern [16], [17] to prove that the Kirchhoff model is the correct asymptotic limit of the three-dimensional model as the thickness of the plate tends to zero. Although the assumption of a smooth domain is implicit in Morgenstern's work, we can easily extend the analysis techniques of [16] to more general situations. In particular, we show here that in a sequence of convex polygonal domains converging to a circle, the relative error of the Kirchhoff model, when compared to the three-dimensional model with *hard* support, is uniformly of order $\mathcal{O}(h^{1/2})$ in the energy norm, where h is the thickness of the plate. Moreover, by similar techniques we show that the gap between Reissner–Mindlin and Kirchhoff models is uniformly of order $\mathcal{O}(h)$ under the same assumptions. Finally, we show that on a smooth domain, the three models are at most $\mathcal{O}(h^{1/2})$ apart. Hence we conclude that the plate paradox must occur in the hard-support models if h is fixed and sufficiently small.

Let us mention that our results are in parallel with recent benchmark calculations [7]. These calculations confirm, in particular, that the error of the Kirchhoff model with respect to the three-dimensional model is primarily due to the assumed hard boundary conditions on simply supported polygonal plates. For example, in the case of a uniformly loaded square plate of thickness h = side length/100, the relative error of the Kirchhoff model in energy norm is approximately 11 percent when compared to the three-dimensional model with soft support, and approximately 2 percent when compared to the hard-support model [7]. This example also shows that the error of the Kirchhoff model may be quite large even for relatively thin plates of simple shape.

The results above show that imposing various boundary conditions that are seemingly close, such as hard and soft simple support, can influence the solution in the entire domain and not only in the boundary layer. Very likely such effects also occur for other boundary conditions for both plates and shells. Therefore, since *any* boundary condition is an idealization of reality, finding the "correct" boundary conditions is an important and sometimes difficult part of building a dimensionally reduced model. For example, both soft and hard simple support can be poor approximations of the real "simple" support.

The plan of the paper is as follows. Section 2 gives the preliminaries and basic formulations of the plate problems. Section 3 elaborates on the variational formulations of the plate problems and presents various energy estimates. Section 4 addresses the problem of the plate paradox. Finally, Appendices A, B, and C present some auxiliary results needed in §§ 3 and 4.

2. Preliminaries. Consider an elastic plate of thickness h that occupies the region $\Omega=\omega\times(-h/2, h/2)$, where $\omega\in\mathbb{R}^2$ is a Lipschitz bounded domain. We assume that the plate is subject to given normal tractions p (i.e., the load) on $\omega\times\{-h/2\}$ and

$\omega \times \{h/2\}$ and that it is simply supported on $\partial\omega \times (-h/2, h/2)$ in such a way that if $\underset{\sim}{u} = (u_1, u_2, u_3)$ is the displacement field, then

$$u_3(x) = 0, \qquad x \in \partial\omega \times \left(-\frac{h}{2}, \frac{h}{2}\right) \tag{2.1}$$

and the other two conditions are natural boundary conditions describing homogenous (zero) components of tractions. Later this condition will be called the soft simple support. If we assume for the moment that no other geometric boundary conditions other than (2.1) are imposed, the plate-bending problem can be formulated as follows. Find the displacement field $\underset{\sim}{u}_0$ that minimizes the quadratic functional of the total energy

$$\begin{aligned} F(\underset{\sim}{u}) &= \frac{1}{2}\int_\Omega \left\{\lambda(\operatorname{div}\underset{\sim}{u})^2 + \mu \sum_{i,j=1}^{3} [\varepsilon_{ij}(\underset{\sim}{u})]^2\right\} dx_1\, dx_2\, dx_3 \\ &\quad - \int_\omega p\frac{1}{2}\left[u_3\left(\cdot, \frac{h}{2}\right) + u_3\left(\cdot, -\frac{h}{2}\right)\right] dx_1\, dx_2 \end{aligned} \tag{2.2}$$

in the Sobolev space $[H^1(\Omega)]^3$ under the boundary condition (2.1). Here $\underset{\approx}{\varepsilon} = \{\varepsilon_{ij}\}_{i,j=1}^3$, $\varepsilon_{ij} = \frac{1}{2}((\partial u_i/\partial x_j) + (\partial u_j/\partial x_i))$ is the strain tensor, and λ and μ are the Lamé coefficients of the material, i.e.,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{1+\nu},$$

where E is the Young modulus and ν is the Poisson ratio, $0 \leqq \nu \leqq \frac{1}{2}$. We also assume that the surface traction p is symmetrically distributed with regard to the planar surfaces of the plate, i.e., we consider a pure bending problem.

So far we have assumed a special model of simple support based on simple geometric constraint (2.1). Of course there are many other possibilities. Later we will discuss another model—of hard simple support—and will discuss the effects of these models of simple support on the solution.

It is well known that if $h/\text{diam}(\omega)$ is small, the three-dimensional plate-bending problem can be formulated in various dimensionally reduced forms (see, e.g., [1], [11], [22]). Here we consider two representatives of such formulations that are used in practice: the Kirchhoff model and the Reissner–Mindlin model (cf. [22] and the references therein).

In general, when ω is fixed and $h \to 0$, the three-dimensional formulation and the dimensionally reduced models converge to the same limit, provided that the load p is appropriately scaled (see below). Hence for sufficiently thin plates the models give practically the same solutions. However, as will be seen later, what is "sufficiently thin" can depend strongly on ω, i.e., the convergence can be very slow in some situations.

In the Kirchhoff model, we approximate the three-dimensional solution as

$$\underset{\sim}{u}_0(x_1, x_2, x_3) \cong \left(-x_3\frac{\partial w_K}{\partial x_1}(x_1, x_2), -x_3\frac{\partial w_K}{\partial x_2}(x_1, x_2), w_K(x_1, x_2)\right),$$

where w_K minimizes the energy

$$F_K(w) = \frac{1}{2}\int_\omega \left\{\nu(\Delta w)^2 + (1-\nu)\sum_{i,j=1}^{2}\left(\frac{\partial^2 w}{\partial x_i \partial x_j}\right)^2\right\} dx_1\, dx_2 - \int_\omega fw\, dx_1\, dx_2 \tag{2.3}$$

in the Sobolev space $H^2(\omega)$ under the boundary condition

$$w = 0 \quad \text{on } \partial\omega. \tag{2.4}$$

Here f is related to p as

$$f=\frac{p}{D}, \qquad D=\frac{Eh^3}{12(1-\nu^2)}. \tag{2.5}$$

When comparing different plate models with fixed ω and variable h we will assume below that f (and not p) is fixed. This ensures that the different models have the same (nontrivial) limit as $h\to 0$. For example, defining the average transverse deflection in the three-dimensional model as

$$w_0=\frac{1}{h}\int_{-h/2}^{h/2} u_3(\cdot, x_3)\,dx_3,$$

we can show (under fairly general assumptions on ω (see [10], [11], [16], [17], and § 3 below) that $\|w_0-w_K\|_{L_2(\omega)}\to 0$ as $h\to 0$.

In the Reissner-Mindlin model, we approximate $\underline{u}_0$ by

$$\underline{u}_0(x_1, x_2, x_3)\approx(-x_3\theta_{R,1}(x_1, x_2), -x_3\theta_{R,2}(x_1, x_2), w_R(x_1, x_2)),$$

where $(w_R, \underline{\theta}_R)$ minimizes the energy

$$\begin{aligned} F_R(w, \underline{\theta})=&\frac{1}{2}\int_\omega \left\{\nu(\operatorname{div}\underline{\theta})^2+(1-\nu)\sum_{i,j=1}^{2}[\varepsilon_{ij}(\underline{\theta})]^2\right\}dx_1\,dx_2 \\ &+\frac{1}{2}\left(\frac{\kappa}{h^2}\right)\int_\omega |\underline{\theta}-\underline{\nabla} w|^2\,dx_1\,dx_2-\int_\omega fw\,dx_1\,dx_2 \end{aligned} \tag{2.6}$$

in the Sobolev space $[H^1(\omega)]^3$ under the boundary condition (2.4). Here $\kappa=6(1-\nu)\kappa_0$, where $\kappa_0=O(1)$ is an additional shear correction factor that may take various values in practice.

We point out that the Kirchhoff approximation to $\underline{u}_0$ satisfies, in addition to (2.1), the boundary condition

$$(u_1t_1+u_2t_2)(x)=0, \qquad x\in\partial\omega\times\left(-\frac{h}{2},\frac{h}{2}\right), \tag{2.7}$$

where $\underline{t}=(t_1, t_2)$ denotes the tangent to $\partial\omega$. This suggests that we should also consider the original plate-bending problem under such more restrictive geometric boundary conditions. Below we refer to the boundary conditions (2.1) and (2.7) and their counterpart in the Reissner-Mindlin model, i.e., (2.4) together with

$$\theta_1t_1+\theta_2t_2=0 \quad \text{on } \partial\omega, \tag{2.8}$$

as hard simple support in contrast to conditions (2.1) and (2.4), which we refer to as soft simple support. Hence, when using the Kirchhoff model we have in mind hard (not soft) simple support. Later we will show that the incapacity of the Kirchhoff model to represent soft simple support can be a severe deficiency of the model on polygonal domains.

3. Variational formulations of the plate-bending problem. Energy estimates. In §§ 3.1-3.3 and in the related Appendix A, we summarize first some basic characteristics of variational formalisms and energy principles associated to the plate-bending problem in its various forms. These results are basically known, but we present them here for the reader's convenience. In § 3.4 we prove some energy estimates relating the Kirchhoff model to both the Reissner-Mindlin model and the three-dimensional model, using the results of the previous subsections.

We assume that the plate occupies the region $\Omega = \omega \times (-h/2, h/2)$, where ω is a Lipschitz bounded domain. Our particular interest is in the cases where ω is either a convex polygon or a smooth domain.

We denote by $H^s(\omega)$, respectively, $H^s(\Omega)$, the usual Sobolev spaces with index $s > 0$. The seminorm and norm of the spaces $[H^s(\omega)]$ or $[H^s(\omega)]^k$ are denoted by $|\cdot|_{s,\omega}$ and $\|\cdot\|_{s,\omega}$, respectively, $|\cdot|_{s,\Omega}$ and $\|\cdot\|_{s,\Omega}$. By $(\cdot,\cdot)$ we mean the inner product of $[L_2(\omega)]^k$ or $[L_2(\Omega)]^k$, and by $\langle\cdot,\cdot\rangle$ the pairing of a space and its dual. The dual space of $H_0^1(\omega)$ will often be needed below and is denoted by $H^{-1}(\omega)$.

3.1. The three-dimensional model. Let us denote by N the space of horizontal rigid displacements of the plate

$$N = \{(\alpha_1 x_1 + \alpha_3 x_2, \alpha_2 x_2 - \alpha_3 x_1, 0),\ \alpha_i \in \mathbb{R},\ i = 1, 2, 3\}.$$

We define the space of (geometrically) admissible displacements in the case of soft simple support as

$$U = \left\{ \underline{u} \in [H^1(\Omega)]^3 \colon u_3 = 0 \text{ on } \partial\omega \times \left(-\frac{h}{2}, \frac{h}{2}\right),\ (\underline{u}, \underline{v}) = 0\ \ \forall \underline{v} \in N \right\} \tag{3.1a}$$

and in the case of hard simple support as

$$U = \left\{ \underline{u} \in [H^1(\Omega)]^3 \colon u_3 = t_1 u_1 + t_2 u_2 = 0 \text{ on } \partial\omega \times \left(-\frac{h}{2}, \frac{h}{2}\right),\ (\underline{u}, \underline{v}) = 0\ \ \forall \underline{v} \in N \right\}. \tag{3.1b}$$

(For simplicity, here we also remove all the horizontal rigid displacements in case of hard support.) Furthermore, we let $\mathscr{H}$ stand for the space of stress or strain tensors defined as

$$\mathscr{H} = \{\underline{\underline{\sigma}} = (\sigma_{ij})_{i,j=1}^3 \colon \sigma_{ij} \in L_2(\Omega),\ \sigma_{ij} = \sigma_{ji}\},$$

and introduce a linear mapping $S \colon \mathscr{H} \to \mathscr{H}$ representing a scaled stress-strain relationship of a linear elastic material:

$$(S\underline{\underline{\tau}})_{i,j} = D^{-1}[\lambda \operatorname{tr}(\underline{\underline{\tau}})\delta_{ij} + \mu\tau_{ij}],$$

where λ and μ are the Lamé coefficients and the scaling factor D is as in (2.5). Then S is one to one and

$$(S^{-1}\underline{\underline{\tau}})_{i,j} = \frac{D}{E}[(1+\nu)\tau_{ij} - \nu \operatorname{tr}(\underline{\underline{\tau}})\delta_{ij}]. \tag{3.2}$$

Moreover, S and S^{-1} are self-adjoint if $\mathscr{H}$ is supplied with the natural inner product

$$(\underline{\underline{\sigma}}, \underline{\underline{\tau}})_{\mathscr{H}} = \sum_{i,j=1}^{3} (\sigma_{ij}, \tau_{ij}).$$

Let us further define the bilinear forms

$$\mathscr{A}(\underline{u}, \underline{v}) = (\underline{\underline{\varepsilon}}(\underline{u}), S\underline{\underline{\varepsilon}}(\underline{v}))_{\mathscr{H}}, \qquad \underline{u}, \underline{v} \in U,$$

$$\mathscr{B}(\underline{\underline{\sigma}}, \underline{\underline{\tau}}) = (\underline{\underline{\sigma}}, S^{-1}\underline{\underline{\tau}})_{\mathscr{H}}, \qquad \underline{\underline{\sigma}}, \underline{\underline{\tau}} \in \mathscr{H},$$

and the linear functional

$$Q(\underline{v}) = \frac{1}{2}\int_\omega f\left[v_3\left(\cdot, \frac{h}{2}\right) + v\left(\cdot, -\frac{h}{2}\right)\right] dx_1\, dx_2,$$

where it is assumed that $f \in L_2(\omega)$, to imply that Q is a bounded linear functional on U (by standard trace inequalities).

In the notation above, the *energy principle* states that the displacement field $\underset{\sim}{u}_0$ due to the load $f=(p/D)\in L_2(\omega)$ is determined as the solution to the following minimization problem. Find $\underset{\sim}{u}_0\in U$ that minimizes in U the functional

$$\mathcal{F}(\underset{\sim}{u})=\tfrac{1}{2}\mathcal{A}(\underset{\sim}{u},\underset{\sim}{u})-Q(\underset{\sim}{u}).$$

The existence and uniqueness of $\underset{\sim}{u}_0$ is due to the following coercivity inequality, known as the Korn inequality (cf. [19]).

LEMMA 3.1. *If U is defined by* (3.1a), *there is a positive constant c such that*

$$\mathcal{A}(\underset{\sim}{u},\underset{\sim}{u})\geqq c\|\underset{\sim}{u}\|_{1,\Omega}^2,\qquad \underset{\sim}{u}\in U. \tag{3.3}$$

We point out that the constant in (3.3) depends on ω (and h), although it is positive for any given Lipschitz domain. In Appendix B we show that the constant in (3.3) remains uniformly positive over a certain family of domains, a result needed in § 5 below.

Given f and the corresponding displacement field $\underset{\sim}{u}_0$, let $\underset{\approx}{\sigma}_0=S\underset{\sim}{u}_0$ be the corresponding (scaled) stress field. The pair $(\underset{\sim}{u},\underset{\approx}{\sigma})=(\underset{\sim}{u}_0,\underset{\approx}{\sigma}_0)$ is then the solution to the following variational problem. Find $(\underset{\sim}{u},\underset{\approx}{\sigma})\in U\times\mathcal{H}$ such that

$$\mathcal{B}(\underset{\approx}{\sigma},\underset{\approx}{\tau})-(\underset{\approx}{\varepsilon}(\underset{\sim}{u}),\underset{\approx}{\tau})_{\mathcal{H}}=0,\qquad \underset{\approx}{\tau}\in\mathcal{H}, \tag{3.4a}$$

$$(\underset{\approx}{\sigma},\underset{\approx}{\varepsilon}(\underset{\sim}{v}))=Q(\underset{\sim}{v}),\qquad \underset{\sim}{v}\in U. \tag{3.4b}$$

It can be easily verified following [5] and [9] (see Appendix A) that the solution to (3.4) exists and is unique.

Finally, we mention that according to the *complementary energy principle*, $\underset{\approx}{\sigma}_0$ is found alternatively as the solution to the following minimization problem [19]. Find $\underset{\approx}{\sigma}_0\in\mathcal{H}$ that minimizes in $\mathcal{H}$ the functional

$$\mathcal{G}(\underset{\approx}{\sigma})=\tfrac{1}{2}\mathcal{B}(\underset{\approx}{\sigma},\underset{\approx}{\sigma})$$

under the constraint (3.4b).

In § 4 below we need the following corollary of the two energy principles (cf. [16]).

LEMMA 3.2. *For any* $(\underset{\sim}{u},\underset{\approx}{\sigma})\in U\times\mathcal{H}$ *such that* $\underset{\approx}{\sigma}$ *satisfies* (3.4b), *the following identity holds*:

$$\tfrac{1}{2}\mathcal{A}(\underset{\sim}{u}_0-\underset{\sim}{u},\underset{\sim}{u}_0-\underset{\sim}{u})+\tfrac{1}{2}\mathcal{B}(\underset{\approx}{\sigma}_0-\underset{\approx}{\sigma},\underset{\approx}{\sigma}_0-\underset{\approx}{\sigma})=\mathcal{F}(\underset{\sim}{u})+\mathcal{G}(\underset{\approx}{\sigma}).$$

Proof. It follows from the energy principle that

$$\mathcal{A}(\underset{\sim}{u}_0,\underset{\sim}{v})=Q(\underset{\sim}{v}),\qquad \underset{\sim}{v}\in U,$$

and from the complementary energy principle that

$$\mathcal{B}(\underset{\approx}{\sigma}_0,\underset{\approx}{\tau})=0,\qquad \underset{\approx}{\tau}\in\mathcal{H}\colon(\underset{\approx}{\tau},\underset{\approx}{\varepsilon}(\underset{\sim}{v}))_{\mathcal{H}}=Q(\underset{\sim}{v})\quad\forall\underset{\sim}{v}\in U.$$

Therefore, in particular, $\mathcal{A}(\underset{\sim}{u}_0,\underset{\sim}{u})=Q(\underset{\sim}{u})$ and $\mathcal{B}(\underset{\approx}{\sigma}_0,\underset{\approx}{\sigma})=\mathcal{B}(\underset{\approx}{\sigma}_0,\underset{\approx}{\sigma}_0)$, and hence

$$\begin{aligned}\tfrac{1}{2}\mathcal{A}(\underset{\sim}{u}_0-\underset{\sim}{u},\underset{\sim}{u}_0-\underset{\sim}{u})+\tfrac{1}{2}\mathcal{B}(\underset{\approx}{\sigma}_0-\underset{\approx}{\sigma},\underset{\approx}{\sigma}_0-\underset{\approx}{\sigma})&=[\tfrac{1}{2}\mathcal{A}(\underset{\sim}{u},\underset{\sim}{u})-\mathcal{A}(\underset{\sim}{u}_0,\underset{\sim}{u})+\tfrac{1}{2}\mathcal{B}(\underset{\approx}{\sigma},\underset{\approx}{\sigma})]\\&\quad+[\tfrac{1}{2}\mathcal{A}(\underset{\sim}{u}_0,\underset{\sim}{u}_0)+\tfrac{1}{2}\mathcal{B}(\underset{\approx}{\sigma}_0,\underset{\approx}{\sigma}_0)-\mathcal{B}(\underset{\approx}{\sigma}_0,\underset{\approx}{\sigma})]\\&=[\mathcal{F}(\underset{\sim}{u})+\mathcal{G}(\underset{\approx}{\sigma})]+[\tfrac{1}{2}\mathcal{A}(\underset{\sim}{u}_0,\underset{\sim}{u}_0)-\tfrac{1}{2}\mathcal{B}(\underset{\approx}{\sigma}_0,\underset{\approx}{\sigma}_0)]\\&=\mathcal{F}(\underset{\sim}{u})+\mathcal{G}(\underset{\approx}{\sigma}).\end{aligned}$$

□

3.2. The Reissner–Mindlin model. In the Reissner–Mindlin model geometrically admissible displacements $(w,\underset{\sim}{\theta})$ span the space $H_0^1(\omega)\times V$, where either

$$V=[H^1(\omega)]^2 \tag{3.5a}$$

or

(3.5b) $$V=\{\underline{\theta}\in[H^1(\omega)]^2\colon t_1\theta_1+t_2\theta_2=0 \text{ on } \partial\omega\}$$

corresponding to soft and hard boundary conditions, respectively. We let $\mathcal{K}$ stand for the space of momentum and curvature tensors:

$$\mathcal{K}=\{\underline{\underline{m}}=(m_{ij})_{i,j=1}^2,\ m_{ij}\in L_2(\omega),\ m_{12}=m_{21}\},$$

and supply $\mathcal{K}$ with the natural inner product

$$(\underline{\underline{m}},\underline{\underline{k}})_{\mathcal{K}}=\sum_{i,j=1}^{2}(m_{ij},k_{ij}).$$

Furthermore, we introduce the linear mapping $T:\mathcal{K}\to\mathcal{K}$ as defined by

$$(T\underline{\underline{k}})_{ij}=\nu\,\mathrm{tr}\,(\underline{\underline{k}})\delta_{ij}+(1-\nu)k_{ij},\qquad \underline{\underline{k}}\in\mathcal{K}.$$

The inverse of T is given by

(3.6) $$(T^{-1}\underline{\underline{k}})_{ij}=\frac{1}{1-\nu}k_{ij}-\frac{\nu}{1+\nu}\mathrm{tr}\,(\underline{\underline{k}})\delta_{ij},$$

and obviously T and T^{-1} are self-adjoint.

Finally we introduce the bilinear forms

$$\mathcal{A}_R(w,\underline{\theta};z,\underline{\varphi})=(\underline{\underline{\varepsilon}}(\underline{\theta}),T\underline{\underline{\varepsilon}}(\underline{\varphi}))+\left(\frac{\kappa}{h^2}\right)(\underline{\theta}-\underline{\nabla}w,\underline{\varphi}-\underline{\nabla}z),\quad w,z\in H_0^1(\omega),\quad \underline{\theta},\underline{\varphi}\in V,$$

$$\mathcal{B}_R(\underline{\underline{m}},\underline{\gamma};\underline{\underline{k}},\underline{\zeta})=(\underline{\underline{m}},T^{-1}\underline{\underline{k}})_{\mathcal{K}}+\left(\frac{h^2}{\kappa}\right)(\underline{\gamma},\underline{\zeta}),\quad \underline{\underline{m}},\underline{\underline{k}}\in\mathcal{K},\quad \underline{\gamma},\underline{\zeta}\in[L_2(\omega)]^2,$$

where $\underline{\underline{\varepsilon}}(\underline{\theta})=(\varepsilon_{ij}(\underline{\theta}))_{i,j=1}^2$ and κ are as in (2.6).

In the notation above, the Reissner–Mindlin formulation of the plate-bending problem, as stated according to the energy principle, is to find the pair $(w_R,\underline{\theta}_R)\in H_0^1(\omega)\times V$ that minimizes in $H_0^1(\omega)\times V$ the functional

$$\mathcal{F}_R(w,\underline{\theta})=\tfrac{1}{2}\mathcal{A}_R(w,\underline{\theta};w,\underline{\theta})-\langle f,w\rangle$$

for a given $f\in H^{-1}(\omega)$.

The existence and uniqueness of $(w_R,\underline{\theta}_R)$ is the consequence of the following lemma, which is proved in Appendix B in slightly more general form (see Lemma B.2 of Appendix B).

LEMMA 3.3. *There is a positive constant c such that*

$$(\underline{\underline{\varepsilon}}(\underline{\theta}),T\underline{\underline{\varepsilon}}(\underline{\theta}))_{\mathcal{K}}+\|\underline{\theta}-\underline{\nabla}w\|_{0,\omega}^2\geqq c(\|\underline{\theta}\|_{1,\omega}^2+\|w\|_{1,\omega}^2),$$
$$\underline{\theta}\in[H^1(\omega)]^2,\quad w\in H_0^1(\omega).$$

Remark 3.1. Regarding the validity of Lemma 3.3 uniformly over a sequence of domains, see Appendix B. (Such a result is needed in § 5 below.)

The analogy of the variational formulation (3.4) is stated for the Reissner–Mindlin model as follows. Find $(w,\underline{\theta},\underline{\underline{m}},\underline{\gamma})\in H_0^1(\omega)\times V\times\mathcal{K}\times[L_2(\omega)]^2$ such that

(3.7a) $$(\underline{\underline{m}},T^{-1}\underline{\underline{k}})_{\mathcal{K}}-(\underline{\underline{\varepsilon}}(\underline{\theta}),\underline{\underline{k}})_{\mathcal{K}}=0,\qquad \underline{\underline{k}}\in\mathcal{K},$$

(3.7b) $$(h^2/\kappa)(\underline{\gamma},\underline{\zeta})-(\underline{\theta}-\underline{\nabla}w,\underline{\zeta})=0,\qquad \underline{\zeta}\in[L_2(\omega)]^2,$$

(3.7c) $$(\underline{\underline{m}},\underline{\underline{\varepsilon}}(\underline{\varphi}))_{\mathcal{K}}+(\underline{\gamma},\underline{\varphi})=0,\qquad \underline{\varphi}\in V,$$

(3.7d) $$-(\underline{\gamma},\underline{\nabla}z)=\langle f,z\rangle,\qquad z\in H_0^1(\omega).$$

The (unique; see Appendix A) solution to this problem is $(w_R, \underline{\theta}_R, \underline{\underline{m}}_R, \underline{\gamma}_R)$, where $\underline{\underline{m}}_R = T\underline{\underline{\varepsilon}}(\underline{\theta}_R)$ and $\underline{\gamma}_R = (\kappa/h^2)(\underline{\theta}_R - \underline{\nabla} w_R)$ have the physical meaning of momentum and (vertical) shear stress field, respectively, both being scaled by a factor D^{-1}.

Finally we note that the pair $(\underline{\underline{m}}_R, \underline{\gamma}_R)$ can be obtained alternatively as the solution to the following minimization problem (the complementary energy principle). Find $(\underline{\underline{m}}_R, \underline{\gamma}_R) \in \mathcal{K} \times [L_2(\omega)]^2$ that minimizes in $\mathcal{K} \times [L_2(\omega)]^2$ the functional

$$\mathcal{G}_R(\underline{\underline{m}}, \underline{\gamma}) = \tfrac{1}{2}\mathcal{B}_R(\underline{\underline{m}}, \underline{\gamma}; \underline{\underline{m}}, \underline{\gamma})$$

under the constraints (3.7c, d).

Upon combining the two energy principles we obtain, in analogy with Lemma 3.2, the following lemma.

LEMMA 3.4. *For any* $(w, \theta) \in H_0^1(\omega) \times V$ *and for any* $(\underline{\underline{m}}, \underline{\gamma}) \in \mathcal{K} \times [L_2(\omega)]^2$ *satisfying* (3.7c, d) *the following identity holds*:

$$\tfrac{1}{2}\mathcal{A}_R(w_R - w, \underline{\theta}_R - \underline{\theta}; w_R - w, \underline{\theta}_R - \underline{\theta}) + \tfrac{1}{2}\mathcal{B}_R(\underline{\underline{m}}_R - \underline{\underline{m}}, \underline{\gamma}_R - \underline{\gamma}; \underline{\underline{m}}_R - \underline{\underline{m}}, \underline{\gamma}_R - \underline{\gamma}) = \mathcal{F}_R(w, \underline{\theta}) + \mathcal{G}_R(\underline{\underline{m}}, \underline{\gamma}).$$

3.3. The Kirchhoff model. Upon introducing the space

$$W = \{z \in H^2(\omega) \colon z = 0 \text{ on } \partial\omega\},$$

we formulate the plate-bending problem according to the Kirchhoff model as follows. Given $f \in W'$ (=dual space of W), find $w_K \in W$ that minimizes in W the energy functional

$$\mathcal{F}_K(w) = \tfrac{1}{2}(\underline{\underline{\varepsilon}}(\underline{\nabla} w), T\underline{\underline{\varepsilon}}(\underline{\nabla} w))_{\mathcal{K}} - \langle f, w\rangle,$$

where T and $(\cdot, \cdot)_{\mathcal{K}}$ are the same as in the Reissner–Mindlin model. The existence and uniqueness of $w_{\mathcal{K}}$ in the consequence of the coercivity inequality

$$(\underline{\underline{\varepsilon}}, (\underline{\nabla} w), T\underline{\underline{\varepsilon}}(\underline{\nabla} w))_{\mathcal{K}} \geqq c\|w\|_{2,\omega}^2, \qquad w \in W,$$

which itself is an easy consequence of Lemma 3.3. Note that $w_{\mathcal{K}}$ is uniquely defined, in particular, if $f \in H^{-1}(\omega)$, and note also that the pair $(w_K, \underline{\theta}_K)$, where $\underline{\theta}_K = \underline{\nabla} w_K$, minimizes the Reissner functional $\mathcal{F}_R$ over the subspace $Z \subset H_0^1(\omega) \times V$ defined by

$$Z = \{(w, \underline{\theta}) \in W \times V \colon \theta = \underline{\nabla} w\}.$$

For the Kirchhoff model, the analogy of the mixed variational formulation (3.7) is the following. Given $f \in W'$, find $(w, \underline{\theta}, \underline{\underline{m}}, \underline{\gamma}) \in W \times V \times \mathcal{K} \times V'$ (where V' is the dual space of V) such that

$$(\underline{\underline{m}}, T^{-1}\underline{\underline{k}})_{\mathcal{K}} - (\underline{\underline{\varepsilon}}(\underline{\theta}), \underline{\underline{k}})_{\mathcal{K}} = 0, \qquad \underline{\underline{k}} \in \mathcal{K}, \tag{3.8a}$$

$$\langle \underline{\theta} - \underline{\nabla} w, \underline{\zeta}\rangle = 0, \qquad \underline{\zeta} \in V', \tag{3.8b}$$

$$(\underline{\underline{m}}, \underline{\underline{\varepsilon}}(\underline{\varphi}))_{\mathcal{K}} + \langle \underline{\gamma}, \underline{\varphi}\rangle = 0, \qquad \underline{\varphi} \in V, \tag{3.8c}$$

$$-\langle \underline{\gamma}, \underline{\nabla} z\rangle = \langle f, z\rangle, \qquad z \in W. \tag{3.8d}$$

LEMMA 3.5. *The variational problem* (3.8) *is well posed and the unique solution is* $(w, \underline{\theta}, \underline{\underline{m}}, \underline{\gamma}) = (w_K, \underline{\theta}_K, \underline{\underline{m}}_K, \underline{\gamma}_K)$, *where* $\underline{\theta}_K = \underline{\nabla} w_K$, $\underline{\underline{m}}_K = T\underline{\underline{\varepsilon}}(\underline{\theta}_K)$, *and* $\underline{\gamma}_K$ *is defined by* (3.8c), *i.e.*,

$$\langle \underline{\gamma}_K, \underline{\varphi}\rangle = -(\underline{\underline{m}}_K, \underline{\underline{\varepsilon}}(\underline{\varphi}))_{\mathcal{K}}, \qquad \underline{\varphi} \in V. \tag{3.9}$$

Proof. If $(w, \underline{\theta}, \underline{\underline{m}}, \underline{\gamma}) = (w_K, \underline{\theta}_K, \underline{\underline{m}}_K, \underline{\gamma}_K)$, (3.8a–c) hold trivially. Moreover, since w_K minimizes $\mathcal{F}_K$ in W, we have $(\underline{\underline{m}}_K, \underline{\underline{\varepsilon}}(\underline{\nabla} z))_{\mathcal{K}} = (\underline{\underline{\varepsilon}}(\underline{\nabla} w_K), T\underline{\underline{\varepsilon}}(\underline{\nabla} z))_{\mathcal{K}} = \langle f, z\rangle$ for all $z \in W$; so, by (3.9), (3.8d) holds as well. The well-posedness is proved in Appendix A. □

Remark 3.2. Note that although w_K, $\underline{\theta}_K$, and $\underline{\underline{m}}_K$ obviously do not depend on the way the space V is defined in (3.5), $\underline{\gamma}_K$ certainly does (see below). Hence in this (somewhat weak) sense the "soft" and "hard" formulations are still separate even in the Kirchhoff model.

We need below the following specific result related to the case where ω is a convex polygon.

LEMMA 3.6. *Let w_K be defined as above assuming that ω is a convex polygon and that $f \in H^{-1}(\omega)$. Furthermore, let $\rho \in H_0^1(\omega)$ and $\psi \in H_0^1(\omega)$ be such that*

$$(\underline{\nabla}\rho, \underline{\nabla}\xi) = (\psi, \xi), \qquad \xi \in H_0^1(\omega), \tag{3.10a}$$

$$(\underline{\nabla}\psi, \underline{\nabla}\xi) = \langle f, \xi\rangle, \qquad \xi \in H_0^1(\omega). \tag{3.10b}$$

Then $\rho = w_K$ and $\psi = -\Delta w_K$.

Proof. From (3.10a, b) it is obvious that $\psi = -\Delta\rho \in H_0^1(\omega)$, so it suffices to show that $\rho = w_K$. First, since $\psi \in H^1(\omega)$ and since ω is a convex polygon, it follows from (3.10a) that $\rho \in H^2(\omega)$ and $\rho \in H^3(\tilde{\omega})$, $\bar{\tilde{\omega}} \subset \bar{\omega} - UA_i$, A_i being the vertices of ω; i.e., $\rho \in W$ (cf. [13]). Moreover, since $\rho = \Delta\rho = 0$ almost everywhere on $\partial\omega$ and since $\partial\omega$ consists of straight-line segments only, it follows that $\partial^2\rho/\partial t^2 = \partial^2\rho/\partial n^2 = 0$ almost everywhere on $\partial\omega$. Therefore, and noting also that $\partial z/\partial t = 0$ almost everywhere on $\partial\omega$ if $z \in W$, integrating by parts shows that

$$\begin{aligned}(\underline{\nabla}\psi, \underline{\nabla}z) &= -(\underline{\nabla}(\Delta\rho), \underline{\nabla}z) = -\nu(\underline{\nabla}(\Delta\rho), \underline{\nabla}z) - (1-\nu)\sum_{i,j=1}^{2}\left(\frac{\partial^3\rho}{\partial x_i\partial^2 x_j^2}, \frac{\partial z}{\partial x_i}\right)\\ &= (\underline{\underline{\varepsilon}}(\underline{\nabla}\rho), T\underline{\underline{\varepsilon}}(\underline{\nabla}z)) - \int_{\partial\omega}\left[\nu\Delta\rho + (1-\nu)\frac{\partial^2\rho}{\partial n^2}\right]\frac{\partial z}{\partial n}\,ds\\ &= (\underline{\underline{\varepsilon}}(\underline{\nabla}\rho), T\underline{\underline{\varepsilon}}(\underline{\nabla}z)), \qquad z \in W.\end{aligned}$$

Hence, by (3.10b), $(\underline{\underline{\varepsilon}}(\underline{\nabla}\rho), T\underline{\underline{\varepsilon}}(\nabla z)) = \langle f, z\rangle$, for all $z \in W$, so ρ minimizes $\mathcal{F}_K$ in W and accordingly, $\rho = w_K$. □

We can now prove the following result that will be needed in the next subsection.

LEMMA 3.7. *Let ω be either a convex polygon or a smooth domain, and let $(w, \underline{\theta}, \underline{\underline{m}}, \underline{\gamma}) = (w_K, \underline{\theta}_K, \underline{\underline{m}}_K, \underline{\gamma}_K) \in W \times V \times \mathcal{H} \times V'$ be the solution to (3.8) for a given $f \in H^{-1}(\omega)$, and with V defined by (3.5b). Then $\underline{\gamma}_K = -\underline{\nabla}(\Delta w_K) \in [L_2(\omega)]^2$ and $(w_K, \underline{\theta}, \underline{\underline{m}}_K, \underline{\gamma}_K)$ is a solution to (3.7) with $h = 0$ in (3.7b). Moreover, if ω is a convex polygon, then $\|\underline{\gamma}_K\|_{0,\omega} = \|f\|_{-1,\omega}$, where*

$$\|f\|_{-1,\omega} = \sup_{z \in H_0^1(\omega)} \frac{\langle f, z\rangle}{|z|_{1,\omega}},$$

and if ω is a smooth domain, then $\|\underline{\gamma}_K\|_{0,\omega} \leqq C\|f\|_{-1,\omega}$, where C depends on ω.

Proof. If $\underline{\gamma}_K \in [L_2(\omega)]^2$ and $f \in H^{-1}(\omega)$, it follows from a simple closure argument that (3.8) remains valid if W is replaced by V and if $\langle\cdot,\cdot\rangle$ on the left side is replaced by $(\cdot,\cdot)$. To prove that $\underline{\gamma}_K = -\underline{\nabla}(\Delta w_K)$, we integrate by parts in (3.9) to obtain

$$\begin{aligned}\langle\underline{\gamma}_K, \underline{\varphi}\rangle = &-\int_\omega \underline{\nabla}(\Delta w_K)\cdot\underline{\varphi}\,dx_1\,dx_2 + \int_{\partial\omega}\left[\nu\Delta w_K + (1-\nu)\frac{\partial^2 w_K}{\partial n^2}\right]\underline{\varphi}\cdot\underline{n}\,ds\\ &+\int_{\partial\omega}(1-\nu)\frac{\partial^2 w_K}{\partial n\partial t}\underline{\varphi}\cdot\underline{t}\,ds, \qquad \underline{\varphi} \in V.\end{aligned}$$

Here the first boundary integral vanishes because $\nu\Delta w + (1-\nu)(\partial^2 w/\partial n^2) = 0$ on $\partial\omega$ is the natural boundary condition associated to the problem of minimizing $\mathcal{F}_K$, and the second boundary integral vanishes since $\underset{=}{\varphi} \cdot \underset{=}{t} = 0$, $\underset{=}{\varphi} \in V$, assuming that V is defined by (3.5b). Hence $\underset{=}{\gamma}_K = -\underset{=}{\nabla}(\Delta w_K)$. On the other hand, from (3.10) we have $\underset{=}{\gamma}_K \in [L_2(\omega)]^2$, and from the well-posedness of (3.8) we see that indeed $\underset{=}{\gamma}_K = -\underset{=}{\nabla}(\Delta w_K)$.

Having verified that $\underset{=}{\gamma}_K = -\underset{=}{\nabla}(\Delta w_K)$, we conclude from Lemma 3.6 that $\underset{=}{\gamma}_K = \underset{=}{\nabla}\psi$, where $\psi \in \overset{\circ}{H}{}^1_0(\omega)$ satisfies (3.10b), so $\|\underset{=}{\gamma}_K\|_{0,\omega} = \|f\|_{-1,\omega}$ as asserted. Finally, if ω is a smooth domain, a standard elliptic regularity estimate implies that $\|\underset{=}{\gamma}_K\|_{0,\omega} \leqq C\|w\|_{3,\omega} \leqq C_1\|f\|_{-1,\omega}$. □

Remark 3.3. It is essential for our results in the next section that when ω is a convex polygon, $\|\underset{=}{\gamma}_K\|_{0,\omega}$ is bounded by $\|f\|_{-1,\omega}$ independently of ω, in contrast to the smooth domain, where the constant depends on ω.

3.4. Energy estimates in case of hard support. Let us define the energy norms

$$|||\underset{=}{u}, \underset{=}{\sigma}|||^2 = \mathscr{A}(\underset{=}{u}, \underset{=}{u}) + \mathscr{B}(\underset{=}{\sigma}, \underset{=}{\sigma}), \qquad (\underset{=}{u}, \underset{=}{\sigma}) \in U \times \mathscr{H},$$

$$|||w, \underset{=}{\theta}, \underset{=}{m}, \underset{=}{\gamma}|||_R^2 = \mathscr{A}_R(w, \underset{=}{\theta}; w, \underset{=}{\theta}) + \mathscr{B}_R(\underset{=}{m}, \gamma; \underset{=}{m}, \underset{=}{\gamma}),$$

$$(w, \underset{=}{\theta}, \underset{=}{m}, \underset{=}{\gamma}) \in \overset{\circ}{H}{}^1_0(\omega) \times V \times \mathscr{H} \times [L_2(\omega)]^2,$$

where the bilinear forms are as defined in §§ 3.1 and 3.2. Then by Lemma 3.2 we have the identity

$$|||\underset{=}{u}_0 - \underset{=}{u}, \underset{=}{\sigma}_0 - \underset{=}{\sigma}|||^2 = |||\underset{=}{u}, \underset{=}{\sigma}|||^2 - 2Q(\underset{=}{u}) \tag{3.11}$$

whenever $\underset{=}{u} \in U$ and $\underset{=}{\sigma} \in \mathscr{H}$ satisfies the constraint (3.4b). Similarly, by Lemma 3.4,

$$|||w_R - w, \underset{=}{\theta}_R - \underset{=}{\theta}, \underset{=}{m}_R - \underset{=}{m}, \underset{=}{\gamma}_R - \underset{=}{\gamma}|||_R^2 = |||w, \underset{=}{\theta}, \underset{=}{m}, \underset{=}{\gamma}|||_R^2 - 2\langle f, w\rangle, \tag{3.12}$$

where $(w, \underset{=}{\theta}) \in \overset{\circ}{H}{}^1_0(\omega) \times V$ and $(\underset{=}{m}, \underset{=}{\gamma}) \in \mathscr{H} \times [L_2(\omega)]^2$ satisfies constraints (3.7c, d).

Let us first apply (3.12) to estimate the gap between the Reissner "quadruple" $(w_R, \underset{=}{\theta}_R, \underset{=}{m}_R, \underset{=}{\gamma}_R)$ and the Kirchhoff "quadruple" $(w_K, \underset{=}{\theta}_K, \underset{=}{m}_K, \underset{=}{\gamma}_K)$. By Lemma 3.7, the choice $(w, \underset{=}{\theta}, \underset{=}{m}, \underset{=}{\gamma}) = (w_K, \underset{=}{\theta}_K, \underset{=}{m}_K, \underset{=}{\gamma}_K)$ is legitimate in (3.12) under the assumptions that ω is either a convex polygon or a smooth domain; $f \in H^{-1}(\omega)$; and V is defined by (3.5b), i.e., the case of hard support. Upon simplifying the right side of (3.12), in this case we obtain the identity

$$|||w_R - w_K, \underset{=}{\theta}_R - \underset{=}{\theta}_K, \underset{=}{m}_R - \underset{=}{m}_K, \underset{=}{\gamma}_R - \underset{=}{\gamma}_K|||_R^2 = (h^2/\kappa)\|\underset{=}{\gamma}_K\|_{0,\omega}^2,$$

which together with Lemma 3.7 leads to the following theorem.

THEOREM 3.1. *Let ω be either* (a) *a convex polygon or* (b) *a smooth domain, let $f \in H^{-1}(\omega)$, and let $(w_R, \underset{=}{\theta}_R, \underset{=}{m}_R, \underset{=}{\gamma}_R)$ and $(w_K, \underset{=}{\theta}_K, \underset{=}{m}_K, \underset{=}{\gamma}_K)$ be the solution to* (3.7) *and* (3.8), *respectively, where V is defined by* (3.5b). *Then in case* (a) *we have the identity*

$$|||w_R - w_K, \underset{=}{\theta}_R - \underset{=}{\theta}_K, \underset{=}{m}_R - \underset{=}{m}_K, \underset{=}{\gamma}_R - \underset{=}{\gamma}_K|||_R^2 = (h^2/\kappa)\|f\|_{-1,\omega}^2,$$

where $\|f\|_{-1,\omega}$ is defined as in Lemma 3.7, *and in case* (b) *the estimate*

$$|||w_R - w_K, \underset{=}{\theta}_R - \underset{=}{\theta}_K, \underset{=}{m}_R - \underset{=}{m}_K, \underset{=}{\gamma}_R - \underset{=}{\gamma}_K|||_R^2 = C(h^2/\kappa)\|f\|_{-1,\omega}^2,$$

where C depends on ω.

Remark 3.4. It is easy to verify that

$$|||w_R - w_K, \underset{=}{\theta}_R - \underset{=}{\theta}_K, \underset{=}{m}_R - \underset{=}{m}_K, \underset{=}{\gamma}_R - \underset{=}{\gamma}_K|||_R^2 \geqq E_K - E_R,$$

where E_R and E_K stand for the total energy of the plate in the Kirchhoff and Reissner-Mindlin models, respectively, i.e.,

$$E_K = \mathscr{F}_R(w_K, \underset{=}{\theta}_K) = -\tfrac{1}{2}\langle f, w_K\rangle, \qquad E_R = \mathscr{F}_R(w_R, \underset{=}{\theta}_R) = -\tfrac{1}{2}\langle f, w_R\rangle.$$

In particular, if ω is a convex polygon, Theorem 3.1 and Lemma 3.6 lead to the relative estimate

$$(E_K - E_R)/E_K \leqq C(\omega, f, \nu)h^2/\kappa_0,$$

where κ_0 is the shear correction factor, and

$$C(\omega, f, \nu) = -\frac{1}{6(1-\nu)} \frac{\int_\omega \Delta w_K f \, dx_1 \, dx_2}{\int_\omega w_K f \, dx_1 \, dx_2}.$$

For example, if ω is the unit square and $f(x) \equiv 1$, then $C(\omega, f, \nu) = 3.440428/(1-\nu)$.

Remark 3.5. In case of soft boundary conditions, constraint (3.7c) is more restrictive and rules out the choice $(\underline{\underline{m}}, \underline{\gamma}) = (\underline{\underline{m}}_K, \underline{\gamma}_K)$ in (3.12). It is still possible to find $(\tilde{\underline{\underline{m}}}_K, \tilde{\underline{\gamma}}_K) \in \mathcal{K} \times [L_2(\omega)]^2$, which is close to $(\tilde{\underline{\underline{m}}}_K, \underline{\gamma}_K)$ away from the boundary and satisfies all the required constraints [16]. With such a construction, it is possible to show that if both f and ω are sufficiently smooth, then

$$|||w_R - w_K, \underline{\theta}_R - \underline{\theta}_K, \underline{\underline{m}}_R - \tilde{\underline{\underline{m}}}_K, \underline{\gamma}_R - \tilde{\underline{\gamma}}_K|||_R^2 \leqq C(\omega, f)h.$$

For other estimates of this type see also [11], the references therein, and [20].

Next, we apply (3.11) to bound the difference between the three-dimensional solution and the Kirchhoff solution. To this end, we need to construct a three-dimensional extension $(\underline{u}_K, \underline{\underline{\sigma}}_K) \in U \times \mathcal{H}$ of the Kirchhoff solution $(w_K, \underline{\theta}_K, \underline{\underline{m}}_K, \underline{\gamma}_K)$. Following [16] we define $\underline{u}_K \in U$ as

$$\underline{u}_K = (-x_3\theta_{K,1}, -x_3\theta_{K,2}, w_K + \tfrac{1}{2}x_3^2\psi), \tag{3.13}$$

and $\underline{\underline{\sigma}}_K \in \mathcal{H}$ as

$$\begin{aligned} \sigma_{K,ij} &= -\alpha x_3 m_{K,ij}, \qquad i, j = 1, 2, \\ \sigma_{K,i3} &= \alpha(\tfrac{1}{2}x_3^2 - \tfrac{1}{8}h^2)\gamma_{K,i}, \qquad i = 1, 2, \\ \sigma_{K,33} &= \alpha(-\tfrac{1}{6}x_3^3 + \tfrac{1}{8}h^2 x_3)f, \end{aligned} \tag{3.14}$$

where $\alpha = 12/h^3$ and $\psi \in H_0^1(\omega)$ is so far unspecified. It is easy to check that $\underline{\underline{\sigma}}_K$ satisfies (3.4b) as far as U is defined by (3.1b), so (3.11) applies with the choice $(\underline{u}, \underline{\underline{\sigma}}) = (\underline{u}_K, \underline{\underline{\sigma}}_K)$ in this case. After a short computation, the right side of (3.11) can then be expressed as

$$\begin{aligned} |||\underline{u}_K, \underline{\underline{\sigma}}_K|||^2 - 2Q(\underline{u}_K) = &\frac{(1-\nu)^2}{1-2\nu} \int_\omega \left(\psi + \frac{\nu}{1-\nu}\Delta w_K\right)^2 dx_1 \, dx_2 \\ &+ \frac{3(1-\nu)}{160} h^2 \int_\omega |\nabla \psi|^2 \, dx_1 \, dx_2 \\ &+ \frac{1}{5(1-\nu)} h^2 \int_\omega (|\underline{\gamma}_K|^2 + \nu \Delta w_K f) \, dx_1 \, dx_2 \\ &+ \frac{17}{1680(1-\nu^2)} h^4 \int_\omega f^2 \, dx_1 \, dx_2 - \frac{1}{4} h^2 \int_\omega \psi f \, dx_1 \, dx_2. \end{aligned}$$

Now if ω is a convex polygon, the choice $\psi = (\nu/(1-\nu))\Delta w_K$ is legitimate and leads—recall also that $\|\gamma_K\|_{0,\omega}^2 = -\int_\omega \Delta w_K f \, dx_1 \, dx_2 = \|f\|_{-1,\omega}^2$ (see Lemmas 3.6 and 3.7)—to the identity

$$|||\underline{u}_K, \underline{\underline{\sigma}}_K|||^2 - 2Q(\underline{u}_K) = \frac{32 + 8\nu + 3\nu^2}{160(1-\nu)} h^2 \|f\|_{-1,\omega}^2 + \frac{17}{1680(1-\nu^2)} h^4 \int_\omega f^2 \, dx_1 \, dx_2.$$

On the other hand, if ω is a smooth domain, we can still find for any $\delta > 0$ a $\psi \in H_0^1(\omega)$ so that

$$\int_\omega \left(\psi - \frac{\nu}{1-\nu}\Delta w_K\right)^2 dx_1\, dx_2 \leqq C\delta\nu^2\|\Delta w_K\|_{1,\omega}^2, \tag{3.15a}$$

$$\int_\omega |\nabla\psi|^2\, dx_1\, dx_2 \leqq C\delta^{-1}\nu^2\|\Delta w_K\|_{1,\omega}^2. \tag{3.15b}$$

Since $\|\Delta w_K\|_{1,\omega} \leqq C(\omega)\|f\|_{-1,\omega}$, we obtain in this case, choosing $\delta = \sqrt{1-2\nu}\, h$, the estimate

$$|||\underline{u}_K, \underline{\underline{\sigma}}_K|||^2 - 2Q(\underline{u}_K) \leqq C(\omega)\frac{\nu^2 h^2}{\sqrt{1-2\nu}}\|f\|_{-1,\omega}^2 + \frac{17h^4}{1680(1-\nu^2)}\|f\|_{0,\omega}^2.$$

We thus conclude the following theorem.

THEOREM 3.2. *Assume that ω is either* (a) *a convex polygon or* (b) *a smooth domain. Let $f \in L_2(\omega)$, let $(\underline{u}_0, \underline{\underline{\sigma}}_0) \in U \times \mathcal{H}$ be the solution to* (3.4) *with U defined by* (3.1b), *and let $(\underline{u}_K, \underline{\underline{\sigma}}_K)$ be defined by* (3.13)–(3.14), *where $(w_K, \underline{\theta}_K, \underline{\underline{m}}_K, \gamma_K) \in W \times V \times \mathcal{K} \times V'$ is the solution to* (3.8) *with V defined by* (3.5b), *and either $\psi = (\nu/(1-\nu))\Delta w_K$ (case* (a)) *or ψ satisfies* (3.15a, b) *with $\delta = \sqrt{1-2\nu}\, h$ (case* (b)). *Then in case* (a) *we have the identity*

$$|||\underline{u}_0 - \underline{u}_K, \underline{\underline{\sigma}}_0 - \underline{\underline{\sigma}}_K|||^2 = C_1(\nu)h^2\|f\|_{-1,\omega}^2 + C_2(\nu)h^4\|f\|_{0,\omega}^2$$

and in case (b) *the estimate*

$$|||\underline{u}_0 - \underline{u}_K, \underline{\underline{\sigma}}_0 - \underline{\underline{\sigma}}_K|||^2 \leqq C(\omega)[C_3(\nu)h + h^2]\|f\|_{-1,\omega}^2 + C_2(\nu)h^4\|f\|_{0,\omega}^2,$$

where $\|f\|_{-1,\omega}$ is defined as in Lemma 3.7 *and*

$$C_1(\nu) = \frac{32 + 8\nu + 3\nu^2}{160(1-\nu)}, \quad C_2(\nu) = \frac{17}{1680(1-\nu^2)}, \quad C_3(\nu) = \frac{\nu^2}{\sqrt{1-2\nu}}.$$

Remark 3.6. In the case of soft boundary conditions it is possible to show that, if ω is smooth and f is sufficiently smooth, then

$$|||\underline{u}_0 - \underline{u}_K, \underline{\underline{\sigma}}_0 - \tilde{\underline{\underline{\sigma}}}_K|||^2 \leqq C(\omega, f)[1 + C_3(\nu)]h,$$

where $\tilde{\underline{\underline{\sigma}}}_K$ is close to $\underline{\underline{\sigma}}_K$ away from the boundary strip $\partial\omega \times (-h/2, h/2)$ [11], [16].

4. The plate paradox. Let $\omega_0 \subset \mathbb{R}^2$ be the unit circular domain with the center at the origin, i.e.,

$$\omega^{[0]} = \{(x_1, x_2): r^2 = x_1^2 + x_2^2 < 1\}.$$

Furthermore, let $\omega^{[n]}$, $n = 1, 2, \cdots$, be the sequence of regular $(n+3)$-polygons such that

$$\bar{\omega}^{[n]} \subset \omega^{[n+1]} \subset \bar{\omega}^{[n+1]} \subset \omega^{[0]},$$

$$\omega^{[n]} \to \omega^{[0]} \quad \text{as } n \to \infty$$

in the sense that for any $x \in \omega^{[0]}$ there is $n(x) > 0$ such that $x \in \omega^{[n]}$ for all $n > n(x)$. Finally, let $\Omega^{[n]} = \omega^{[n]} \times (-h/2, h/2)$ and $\Omega^{[0]} = \omega^{[0]} \times (-h/2, h/2)$.

Assume now that the unit load is imposed, i.e., $f = p/D = 1$ (see § 2). Then for fixed thickness h there exist the unique solutions $\underline{u}_0^{[n]}$, $(w_R^{[n]}, \underline{\theta}_R^{[n]})$, and $w_K^{[n]}$, $n = 0, 1, 2, \cdots$, corresponding, respectively, to the three-dimensional, Reissner–Mindlin, and Kirchhoff formulations of the plate-bending problem with either hard or soft simple support. In § 4.1 we will show that $w_K^{[n]} \to w_K^{[\infty]} \neq w_K^{[0]}$ and give explicit expressions for $w_K^{[\infty]}$ and $w_K^{[0]}$. This is the plate paradox in the Kirchhoff model pointed out in [7] and [3]. In § 4.2 we will show that this paradox also occurs in the Reissner-Mindlin model and in the three-dimensional formulation in case of *hard simple support.* Finally, in § 4.3 we show that the paradox *does not occur* in the Reissner-Mindlin and three-dimensional formulations where soft simple support is imposed. This has been briefly noted in [3].

The results clearly show that seemingly minor changes in the boundary conditions can lead to a significant change of the solution on $\Omega^{[n]}$, respectively, $\omega^{[n]}$, when n is large. In fact, we will see that there can be significant changes already when $n=1$.

The main question we will address below in this section is whether, as $n \to \infty$,

$\underline{u}^{[n]} \to \underline{u}^{[0]}$ for the three-dimensional formulation,

$(w_R^{[n]}, \underline{\theta}_R^{[n]}) \to (w_R^{[0]}, \underline{\theta}_R^{[0]})$ for the Reissner-Mindlin model,

$w_k^{[n]} \to w_k^{[0]}$ for the Kirchhoff model.

4.1. The plate paradox for the Kirchhoff model. We have shown in Lemma 3.6 that for $n=1,2,\cdots$, $w_k^{[n]}=\rho^{[n]}$, and $-\Delta w_k^{[n]}=\psi^{[n]}$, where $\rho^{[n]}$ and $\psi^{[n]}$ satisfy (3.10a, b).

THEOREM 4.1. *Let* $\rho^{[\infty]}, \psi^{[\infty]} \in H_0^1(\omega^{[0]})$ *be such that*

$$(\underline{\nabla}\rho^{[\infty]}, \underline{\nabla}\xi) = (\psi^{[\infty]}, \xi), \qquad \xi \in H_0^1(\omega^{[0]}), \tag{4.1a}$$

$$(\underline{\nabla}\psi^{[\infty]}, \underline{\nabla}\xi) = \langle f, \xi\rangle, \qquad \xi \in H_0^1(\omega^{[0]}) \tag{4.1b}$$

with $f=1$. *Then as* $n \to \infty$

$$\|\psi^{[n]} - \psi^{[\infty]}\|_{H^1(\omega^{[0]})} \to 0, \qquad \|\varphi^{[n]} - \rho^{[\infty]}\|_{H^1(\omega^{[0]})} \to 0.$$

Here we understand that $\psi^{[n]}$ *and* $\varphi^{[n]}$ *extend by zero from* $\omega^{[n]}$ *to* $\omega^{[0]}$.

Proof. Let P_n denote the orthogonal projection of $H_0^1(\omega^{[0]})$ onto the subspace $H_0^{1,n}(\omega^{[0]})$ defined by

$$H^{1,n}(\omega^{[0]}) = \{u \in H_0^1(\omega^{[0]}) \colon u = 0 \text{ on } \omega^{[0]} - \omega^{[n]}\}$$

and let $\hat{\psi}^{[n]}$ and $\hat{\rho}^{[n]}$ denote the extension of $\psi^{[n]}$ and $\rho^{[n]}$, respectively, by zero onto $\omega^{[0]}$. Then $\hat{\psi}^{[n]} = P_n\psi^{[\infty]}$ by (4.1b). From Theorem C.1 in Appendix C, it then follows immediately that $\hat{\psi}^{[n]} \to \psi^{[\infty]}$ in $H_0^1(\omega^{[0]})$. From (4.1a) we then see that $\rho^{[n]} - P_n\rho^{[\infty]} \to 0$ in $H_0^1(\omega^{[0]})$, and therefore, by the same argument, that $\hat{\rho}^{[n]} \to \rho^{[\infty]}$ in $H_0^1(\omega^{[0]})$. □

Let us now characterize $\hat{\rho}^{[0]} = w_K^{[0]}$ and $\rho^{[\infty]} = w_K^{[\infty]}$ more explicitly. To this end, note first that $\rho^{[\infty]}$ is the solution to the problem

$$\Delta\Delta\rho^{[\infty]} = 1 \qquad \text{on } \omega^{[0]}, \tag{4.2a}$$

$$\rho^{[\infty]} = \Delta\rho^{[\infty]} = 0 \quad \text{on } \partial\omega^{[0]}. \tag{4.2b}$$

On the other hand, it is easy to see that $\rho^{[0]}$ is the solution of the problem

$$\Delta\Delta\rho^{[0]} = 1 \quad \text{on } \omega^{[0]}, \tag{4.3a}$$

$$\rho^{[0]} = 0 \qquad \text{on } \partial\omega^{[0]}, \tag{4.3b}$$

$$\nu\Delta\rho^{[0]} + (1-\nu)\frac{\partial^2\rho^{[0]}}{\partial n^2} = 0. \tag{4.3c}$$

Here (4.3c) is the standard boundary condition for the simply supported circular plate (see, e.g., [24, p. 554]). Solutions (4.2) and (4.3) show that

$$\rho^{[\infty]} = C_1^{[\infty]} + C_2^{[\infty]} r^2 + C_3^{[\infty]} r^4, \qquad \rho^{[0]} = C_1^{[0]} + C_2^{[0]} r^2 + C_3^{[0]} r^4,$$

where $r^2 = x_1^2 + x_2^2$,

$$C_3^{[\infty]} = C_3^{[0]} = \frac{1}{64},$$

and C_1, C_2 are determined from the boundary conditions. By simple computation we get

$$\rho^{[0]}(0,0) = w_K^{[0]}(0,0) = \frac{1}{64}\frac{5+\nu}{1+\nu}, \tag{4.4a}$$

$$\rho^{[\infty]}(0,0) = w_K^{[\infty]}(0,0) = \frac{3}{64}, \tag{4.4b}$$

and hence for $\nu = 0.3$ we have

$$\frac{w_K^{[0]}(0,0)}{w_K^{[\infty]}(0,0)} = 1.36,$$

i.e., the gap between $w_K^{[0]}$ and $w_K^{[\infty]}$ is 36 percent at the origin. Analogously, for $\nu = 0.3$,

$$\frac{\| w_K^{[0]} - w_K^{[\infty]} \|_{0,\omega^{[0]}}}{\| w_K^{[\infty]} \|_{0,\omega^{[0]}}} = 0.287.$$

Remark 4.1. We have assumed that $\omega^{[n]}$ were regular polygons. As the proof shows, (4.5b) also holds when $\{\omega^{[n]}\}$ is an arbitrary sequence of *convex* polygons such that $\omega^{[n]} \to \omega^{[0]}$ in the sense described above.

It is essential, however, that $\omega^{[n]}$ are convex polygons. If we replace $\omega^{[n]}$ by $\hat{\omega}^{[n]}$, where $\hat{\omega}^{[n]}$ are nonconvex polygons as shown in Fig. 4.1, then [15] shows that $\hat{\omega}^{[\infty]}$

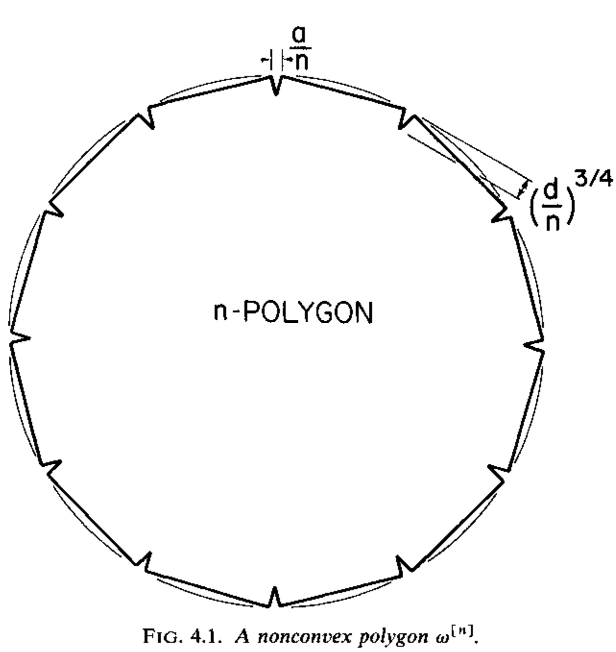


FIG. 4.1. *A nonconvex polygon* $\omega^{[n]}$.

satisfies

$$\Delta\Delta\hat{\omega}^{[\infty]}=1 \qquad \text{in } \omega^{[0]},$$
$$\hat{\omega}^{[\infty]}=\frac{\partial\hat{\omega}^{[\infty]}}{\partial n}=0 \qquad \text{in } \partial\omega^{[0]},$$

and hence

$$\hat{\omega}^{[\infty]}(0,0)=\frac{1}{64}. \tag{4.4c}$$

4.2. The plate paradox for the three-dimensional and Reissner–Mindlin models. We will analyze in detail the case of Reissner–Mindlin model only. The case of the three-dimensional formulation can be dealt with analogously.

THEOREM 4.2. *Let h be fixed and sufficiently small, and let $w_R^{[n]}$ be the Reissner–Mindlin solution on $\omega^{[n]}$ corresponding to unit load $f=1$ on $\omega^{[n]}$ and hard simple support on $\partial\omega^{[n]}$, $n=0,1,2,\cdots$. Then if $w_R^{[n]}$ is extended by zero onto $\omega^{[0]}$, we have*

$$\|w_R^{[n]}-\omega_R^{[0]}\|_{1,\omega^{[0]}}\geqq\alpha>0,$$
$$\left|\int_{\omega^{[0]}}(w_R^{[n]}-w_R^{[0]})\,dx_1\,dx_2\right|\geqq\alpha>0$$

for all $n\geqq n_0$, n_0 large enough.

Proof. By Theorem 3.1 we have

$$|||w_R^{[n]}-w_K^{[n]},\underline{\theta}_R^{[n]}-\underline{\theta}_K^{[n]},\underline{\underline{m}}_R^{[n]}-\underline{\underline{m}}_K^{[n]},\underline{\gamma}_R^{[n]}-\underline{\gamma}_K^{[n]}|||_R^2\leqq h^2/\kappa\|f\|_{-1,\omega^{[n]}}^2,$$
$$|||w_R^{[0]}-w_K^{[0]},\underline{\theta}_R^{[0]}-\underline{\theta}_K^{[0]},\underline{\underline{m}}_R^{[0]}-\underline{\underline{m}}_K^{[0]},\underline{\gamma}_R^{[0]}-\underline{\gamma}_K^{[0]}|||_R^2\leqq Ch^2/\kappa\|f\|_{-1,\omega^{[0]}}^2.$$

Note that $\|f\|_{-1,\omega^{[n]}}\leqq C_0$ independently of n. Using Lemma 3.3 and Theorem B.3, we see that

$$[\|w_R^{[n]}-w_K^{[n]}\|_{1,\omega^{[n]}}^2+\|\underline{\theta}_R^{[n]}-\underline{\theta}_K^{[n]}\|_{1,\omega^{[n]}}^2]\leqq Ch^2,$$
$$[\|w_R^{[0]}-w_K^{[0]}\|_{1,\omega^{[0]}}^2+\|\underline{\theta}_R^{[0]}-\underline{\theta}_K^{[0]}\|_{1,\omega^{[0]}}^2]\leqq Ch^2,$$

where C is independent of n and h. On the other hand, we have by Theorem 4.1,

$$\|w_K^{[n]}-w_K^{[\infty]}\|_{1,\omega^{[0]}}\to 0 \quad \text{as } n\to\infty,$$
$$\|w_K^{[\infty]}-w_K^{[0]}\|_{1,\omega^{[0]}}>0.$$

This shows that for sufficiently small h there is $\alpha>0$ such that $\|w_R^{[0]}-w_R^{[n]}\|_{1,\omega^{[n]}}\geqq\alpha>0$ for all $n>n_0$.

Realizing that (in our case for $f=1$)

$$E_R^{[n]}=-\int_{\omega^{[n]}}w_R^{[n]}\,dx_1\,dx_2, \qquad E_R^{[0]}=-\int w_R^{[0]}\,dx_1\,dx_2,$$
$$E_K^{[n]}=-\int w_K^{[n]}\,dx_1\,dx_2, \qquad E_R^{[0]}=-\int w_K^{[0]}\,dx_1\,dx_2,$$

we also have

$$\left|\int\!\!\int(\omega_R^{[0]}-w_R^{[n]})\,dx_1\,dx_2\right|\geqq\alpha>0 \quad \text{as } n>n_0. \qquad \square$$

Using Theorem 3.2 and analogous arguments, we get Theorem 4.3.

THEOREM 4.3. *Let h be fixed and sufficiently small and let $\underline{u}_0^{[n]}=(u_{01}^{[n]},u_{02}^{[n]},u_{03}^{[n]})$ be the three-dimensional solution of the plate-bending problem on $\Omega^{[n]}$ corresponding to* *the load $p = D$ and hard simple support, $n = 0, 1, 2, \cdots$. Then if $u_{03}^{[n]}$ is extended by zero onto $\Omega^{[0]}$, we have*

$$\|u_{03}^{[n]} - u_{03}^{[0]}\|_{1,\Omega^{[0]}} \geqq \alpha > 0,$$

$$\left| \int_{\omega^{[0]}} \left(u_{03}^{[n]}\left(x_1, x_2, \frac{h}{2}\right) + u_{03}^{[0]}\left(x_1, x_2, \frac{h}{2}\right)\right) dx_1\, dx_2 \right| \geqq \alpha > 0$$

for all $n \geqq n_0$, n_0 sufficiently large.

Theorems 4.2 and 4.3 show that hard simple support leads not only to the paradox in the Kirchhoff model but also to those in the three-dimensional formulation and the Reissner–Mindlin model. (In § 4.3 we will show that the paradox occurs neither in the three-dimensional formulation nor in the Reissner–Mindlin model when the simple soft support is imposed.)

The proof employs the fact that the Kirchhoff model approximates very well the Reissner–Mindlin and three-dimensional formulations for the *hard* support. This shows that the circular plate and polygonal plate solutions are far apart in the entire region and not only in the area close to the boundary, where boundary layer effects occur.

The results above show that plausibly unimportant changes in the boundary conditions could lead to significant changes in the solution through the entire region even if the three-dimensional linear elasticity model is used. We expect that the paradox will also occur in nonlinear formulations. For engineering implications of effects of this type we refer to [6].

4.3. The "nonparadox" in case of soft simple support. In this section we will prove that, in contrast to hard simple support, the solution on $\omega^{[n]}$ converges to the solution on $\omega^{[0]}$ for both the Reissner–Mindlin and the three-dimensional plate model. This is in obvious contrast to hard simple support. We will elaborate in detail on the case of the Reissner–Mindlin model. The analysis of the three-dimensional model is analogous.

Let us denote

$$\mathcal{D}_n = \omega^{[n+1]} - \omega^{[n]}, \qquad n = 1, 2, \cdots,$$
$$\mathcal{D}_0 = \omega^{[1]},$$
$$\mathcal{D}_n^0 = \omega^{[0]} - \omega^{[n]}, \qquad n = 1, 2, \cdots$$

(see Fig. 4.2).

Let $L = (L_2(\omega^{[0]})^3$, $\underline{u} = (w, \underline{\theta}) \in L$ and

$$\mathcal{S}_0 = \{\underline{u} \in L\colon w \in H_0^1(\omega^{[0]}),\ \underline{\theta} \in (H^1(\omega^{[0]}))^2\},$$
$$\mathcal{S}_n = \{u \in L\colon w \in H_0^1(\omega^{[0]}),\ w = 0 \text{ on } \mathcal{D}_n^0,\ \underline{\theta} \in (H^1(\omega^{[0]})^2)\},$$
$$\mathcal{T}_n = \{u \in L\colon w \in H_0^1(\omega^{[0]}),\ \underline{\theta} \in (H^1(\omega^{[n]}))^2,\ \underline{\theta} \in (H^1(\mathcal{D}_m))^2,\ m = n, n+1, \cdots\},$$
$$\mathcal{L}_{n,m} = \{u \in L\colon w \in H_0^1(\omega^{[0]}),\ w = 0 \text{ on } \mathcal{D}_n^0,$$
$$\underline{\theta} \in (H^1(\omega^{[m]}))^2,\ \underline{\theta} \in (H^1(\mathcal{D}_j))^2,\ j = m, m+1, \cdots\}.$$

We have $\mathcal{S}_n \subset \mathcal{S}_0$, $\mathcal{S}_0 \subset \mathcal{T}_n$, and

$$\mathcal{L}_{n,m} \supset \mathcal{S}_n, \qquad \mathcal{L}_{n,m} \subset \mathcal{T}_m.$$

All the spaces are embedded in $\mathcal{T}_1$. Furthermore, let

$$Z_n = \{u \in L\colon w \in H^1(\omega^{[n]}),\ \underline{\theta} \in (H^1(\omega^{[n]})^2\},$$
$$\mathring{Z}_n = \{u \in Z_n\colon w \in H_0^1(\omega^{[n]})\},$$
$$\mathcal{A}_R^0(w, \underline{\theta}; z, \underline{\varphi}) = \sum_{i=0}^{\infty} \mathcal{A}_R^{\mathcal{D}_i}(\underline{u}, \underline{v}),$$

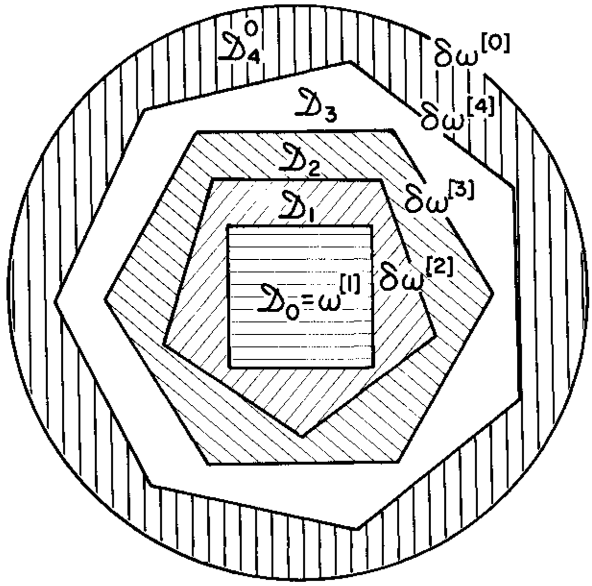


FIG. 4.2. *The configuration of the domains* $\mathcal{D}_n$, $\mathcal{D}_0$, $\mathcal{D}'_n$.

where $\mathcal{A}_R^\omega$ is given in § 3.2 for the region ω and $\mathcal{A}_R^{\mathcal{D}_i}$ has the same form but is integrated only over $\mathcal{D}^i$. Analogously we define $\mathcal{A}_R^{\omega^{[n]}}$, etc. Finally, we supply $\mathcal{T}_1$ with the norm

$$\|\underline{u}\|^2=\sum_{i=0}^{\infty}\mathcal{A}_R^{\mathcal{D}_i}(\underline{u},\underline{u}).$$

To see that $\|\cdot\|$ is indeed a norm, assume that $\underline{u}=(w,\underline{\theta})\in\mathcal{T}_1$ and $\|\underline{u}\|=0$. Then, since the first term in the expression for $\mathcal{A}_R^{\mathcal{D}_i}$ is the same as in the case of plane elasticity (where θ_1, θ_2 play the role of the displacements), we have on $\mathcal{D}_j$, $\theta_1=a_j+c_jx_2$, $\theta_2=b_j-c_jx_1$, and because $\|\underline{\theta}-\nabla w\|_{0,\mathcal{D}_j}=0$ we get $c_j=0$. Hence $w=d_j+a_jx_1+b_jx_2$ on $\mathcal{D}_j$, and so, because $w\in H_0^1(\omega^{[0]})$ we get $w=0$ and $a_j=b_j=0$, $j=0,1,2,\cdots$ (see also Appendix B). Hence $\underline{u}=0$ and accordingly, $\|\cdot\|$ is a norm on $\mathcal{T}_1$.

For $\underline{u}\in Z_n$ let $\|\underline{u}\|^2_{R,\omega^{[n]}}=\mathcal{A}_R^{\omega^{[n]}}(\underline{u},\underline{u})$. Then by Theorem B.1 in Appendix B,

$$\inf_{abc}\|\theta_1-(a+cx_2),\theta_2-(b-cx_1)\|_{1,\omega^{[n]}}\leqq C_n\|u\|_{R,\omega^{[n]}},\tag{4.5a}$$

$$\inf_{abcd}\|w-(d+ax_1+bx_2+cx_1x_2)\|_{1,\omega^{[n]}}\leqq C_n\|u\|_{R,\omega^{[n]}}.\tag{4.5b}$$

Here C_n depends in general on $\omega^{[n]}$.

Assume now that for an $n_0>0$

(4.6a) f has compact support in $\omega^{[n_0]}$,

$$\int_{\omega^{[n_0]}}f\,dx_1\,dx_2=\int_{\omega^{[n_0]}}fx_1\,dx_1\,dx_2=\int_{\omega^{[n_0]}}fx_2\,dx_1\,dx_2=\int_{\omega^{[n_0]}}fx_1x_2\,dx_1\,dx_2=0\tag{4.6b}$$

and that $n>n_0$, $m>n_0$. Then for $\underline{u}\in\mathcal{T}_n$, $n\geqq n_0$,

$$\left|\int_{\omega^{[0]}}fw\,dx_1\,dx_2\right|=\left|\int_{\omega^{[n_0]}}fw\,dx_1\,dx_2\right|\leqq C_{n_0}\|u\|.$$

Hence for $n, m \geqq n_0$ there exist unique

$$\underline{u}(\mathcal{S}_n) \in \mathcal{S}_n, \quad \underline{u}(\mathcal{T}_n) \in \mathcal{T}_n, \quad \underline{u}(\mathcal{L}_{n,m}) \in \mathcal{L}_{n,m}, \quad \underline{u}(\mathring{Z}_n) \in \mathring{Z}_m$$

such that

$$t_R^0(\underline{u}(\mathcal{S}_n), \underline{v}) = \int_{\omega^{[0]}} fz\, dx_1\, dx_2 \quad \forall v \in (z, \underline{\varphi}) \in \mathcal{S}_n$$

and analogously for $u(\mathcal{T}_n)$, $u(\mathcal{L}_{n,m})$, $u(\mathring{Z}_n)$. Obviously $\underline{u}(\mathcal{S}_0) = \underline{u}_R^{[0]}$ and $\underline{u}(\mathring{Z}_n) = \underline{u}_R^{[n]}$, and $\underline{u}(\mathcal{L}_{n,m}) = \underline{u}(\mathring{Z}_n)$ on $\omega^{[n]}$ and is zero on $\mathcal{D}_n^0$.

Using Theorem C.1 we get

(4.7a) $$\underline{u}_R^{[0]} = \underline{u}(\mathcal{S}_0) = \underline{u}(\mathcal{S}_n) + \underline{\rho}(\mathcal{S}_0, \mathcal{S}_n),$$

(4.7b) $$\|\underline{u}(\mathcal{S}_0)\|^2 = \|\underline{u}(\mathcal{S}_n)\|^2 + \|\underline{\rho}(\mathcal{S}_0, \mathcal{S}_n)\|^2,$$

(4.7c) $$\|\underline{\rho}(\mathcal{S}_0, \mathcal{S}_n)\| \to 0 \quad \text{as } n \to \infty;$$

(4.8a) $$\underline{u}(\mathcal{T}_n) = \underline{u}(\mathcal{S}_0) + \underline{\rho}(\mathcal{T}_n, \mathcal{S}_0),$$

(4.8b) $$\|\underline{u}(\mathcal{T}_n)\|^2 = \|\underline{u}(\mathcal{S}_0)\|^2 + \|\underline{\rho}(\mathcal{T}_n, \mathcal{S}_0)\|^2,$$

(4.8c) $$\|\underline{\rho}(\mathcal{T}_n, \mathcal{S}_0)\| \to 0 \quad \text{as } n \to \infty;$$

(4.9a) $$\underline{u}(\mathcal{L}_{n,m}) = \underline{u}(\mathcal{S}_n) + \underline{\rho}(\mathcal{L}_{n,m}, \mathcal{S}_n),$$

(4.9b) $$\|\underline{u}(\mathcal{L}_{n,m})\|^2 = \|\underline{u}(\mathcal{S}_n)\|^2 + \|\underline{\rho}(\mathcal{L}_{n,m}, \mathcal{S}_n)\|^2,$$

(4.9c) $$\|\underline{\rho}(\mathcal{L}_{n,m}, \mathcal{S}_n)\| \to 0 \quad \text{as } m \to \infty;$$

(4.10a) $$\underline{u}(\mathcal{T}_m) = \underline{u}(\mathcal{L}_{n,m}) + \underline{\rho}(\mathcal{T}_m, \mathcal{L}_{n,m}),$$

(4.10b) $$\|\underline{u}(\mathcal{T}_m)\|^2 = \|\underline{u}(\mathcal{L}_{n,m})\|^2 + \|\underline{\rho}(\mathcal{T}_m, \mathcal{L}_{n,m})\|^2,$$

(4.10c) $$\|\underline{\rho}(\mathcal{T}_m, \mathcal{L}_{n,m})\| \to 0 \quad \text{as } n \to \infty.$$

Now let $\varepsilon > 0$ and $n > \max(n(\varepsilon), n_0)$. Then we have

$$\|\underline{\rho}(\mathcal{S}_0, \mathcal{S}_n)\|^2 < \varepsilon, \qquad \|\underline{\rho}(\mathcal{T}_n, \mathcal{S}_0)\|^2 < \varepsilon.$$

Using (4.7)-(4.10) we get

$$\begin{aligned}\|\underline{u}(\mathcal{S}_0)\|^2 + \|\underline{\rho}(\mathcal{T}_m, \mathcal{S}_0)\|^2 = \|\underline{u}(\mathcal{T}_m)\|^2 &= \|\underline{u}(\mathcal{L}_{n,m})\|^2 + \|\underline{\rho}(\mathcal{T}_m, \mathcal{L}_{n,m})\|^2 \\ &= \|\underline{u}(\mathcal{S}_n)\|^2 + \|\underline{\rho}(\mathcal{L}_{n,m}, \mathcal{S}_n)\|^2 + \|\underline{\rho}(\mathcal{T}_n, \mathcal{L}_{n-n})\|^2 \\ &= \|\underline{u}(\mathcal{S}_n)\|^2 - \|\underline{\rho}(\mathcal{S}_0, \mathcal{S}_n)\|^2 + \|\underline{\rho}(\mathcal{L}_{n,m}, \mathcal{S}_n)\|^2 \\ &\quad + \|\underline{\rho}(\mathcal{T}_m, \mathcal{L}_{n,m})\|^2\end{aligned}$$

and hence for $n, m \geqq \max(n(\varepsilon), n_0)$

$$\|\underline{\rho}(\mathcal{T}_m, \mathcal{S}_0)\|^2 + \|\underline{\rho}(\mathcal{S}_0, \mathcal{S}_n)\|^2 = \|\underline{\rho}(\mathcal{L}_{n,m}, \mathcal{S}_n)\|^2 + \|\underline{\rho}(\mathcal{T}_m, \mathcal{L}_{n,m})\|^2 \leqq 2\varepsilon,$$

which yields

$$\|\underline{\rho}(\mathcal{L}_{n,n}, \mathcal{S}_n)\|^2 \leqq 2\varepsilon.$$

Therefore

$$\underline{u}(\mathcal{S}_0) - \underline{u}(\mathcal{L}_{n,n}) = \underline{u}(\mathcal{S}_0) - \underline{u}(\mathcal{S}_n) + \underline{u}(\mathcal{S}_n) - \underline{u}(\mathcal{L}_{n,n}) = \underline{\rho}(\mathcal{S}_0, \mathcal{S}_n) - \underline{\rho}(\mathcal{L}_{n,n}, \mathcal{S}_n)$$

and hence

$$\|\underline{u}(\mathcal{S}_0) - \underline{u}(\mathcal{L}_{n,n})\| \leqq \varepsilon^{1/2} + \sqrt{2\varepsilon} \leqq C\varepsilon^{1/2}.$$

Because, as above, $\underline{u}(\mathcal{L}_{n,n}) = \underline{u}_R^{[n]}$ on $\omega^{[n]}$, and zero on $\mathcal{D}_n^0$, $\underline{u}_R^{[n]} \to \underline{u}_R^{[0]}$ in the space $\mathcal{T}_1$ or in any $\mathcal{T}_m$ for m fixed.

Remark 4.2. Note that until now we have not used Theorem B.3 (Appendix B), but only Theorem B.1.

So far we have assumed that f satisfies (4.6). Let us now study the general case. Assume that $f \in L_2(\omega^{[0]})$.

Let us first note that if $\underline{u}=(w, \theta) \in \mathscr{L}_{n,n}$, then $w \in H_0^1(\omega^{[0]})$ and

$$\|w\|_{1,\omega^{[n]}} = \|w\|_{1,\omega^{[n]}} \leqq C\|\underline{u}\| \tag{4.11}$$

with C independent of n because of Theorem B.3.

For $0<\Delta<\frac{1}{2}$ we denote

$$R_\Delta = \{(x_1, x_2): x_1^2+x_2^2>1-\Delta\},$$
$$\partial R_\Delta = \{(x_1, x_2): x_1^2+x_2^2=1-\Delta\}.$$

Then

$$\|w\|_{0,R_\Delta} \leqq C\Delta\|w\|_{1,\omega^{[0]}} \leqq C\Delta\|u\|,$$
$$\|w\|_{0,\partial R_\Delta} \leqq C\Delta^{1/2}\|w\|_{1,\omega^{[0]}} \leqq C\Delta^{1/2}\|u\|.$$

Now let

$$f_\Delta = \begin{cases} f & \text{on } R_\Delta, \\ 0 & \text{on } \omega^{[0]}-R_\Delta, \end{cases}$$
$$g_\Delta = (a+bx_1+cx_2+dx_1x_2)\mathcal{O}_\Delta,$$

where $\mathcal{O}_\Delta$ is the Dirac function concentrated on ∂R_Δ and a, b, c, d are such that $f_\Delta + g_\Delta$ satisfies (4.6).

For $n>n_{1,\Delta}$ such that $\bar{\bar{R}}_\Delta \subset \omega^{[n_{1,\Delta}]}$, let $\underline{u}_\Delta(\mathscr{L}_{n,n})$ and $\underline{u}_\Delta(\mathscr{S}_0)$ be the solutions when instead of f the function f_Δ is used. Then we get

$$\|\underline{u}_\Delta(\mathscr{L}_{n,n}) - \underline{u}(\mathscr{L}_{n,n})\| \leqq C\Delta^{1/2},$$
$$\|\underline{u}_\Delta(\mathscr{S}_0) - \underline{u}(\mathscr{S}_0)\| \leqq C\Delta^{1/2},$$

where C is independent of n and Δ but, in general, depends on f. Hence we can select Δ so that $C\Delta^{1/2}<\varepsilon$. Furthermore, we have shown

$$\|\underline{u}_\Delta(\mathscr{L}_{n,n}) - \underline{u}_\Delta(\mathscr{S}_0)\| < \varepsilon$$

for all $n \geqq n_1(\varepsilon)$ and therefore

$$\|\underline{u}(\mathscr{L}_{n,n}) - \underline{u}(\mathscr{S}_0)\| < 3\varepsilon$$

for all $n \geqq n_1(\varepsilon)$. Since $\underline{u}(\mathscr{S}_0) = \underline{u}_R^{[0]}$ and $\underline{u}(\mathscr{L}_{n,n}) = \underline{u}_R^{[n]}$, we get

$$\|\underline{u}_R^{[0]} - \underline{u}_R^{[n]}\| \to 0 \quad \text{as } n \to \infty.$$

Here $\underline{u}_R^{[n]} = (w_R^{[n]}, \underline{\theta}_R^{[n]})$ is understood to be extended by zero on $\mathscr{D}_n^0$ and $\|\cdot\|$ is the norm in $\mathscr{T}_1$ (note that $w_R^{[n]} \in H_0^1(\omega^{[0]})$, but $\underline{\theta}_R^{[n]} \not\in H^1(\omega^{[0]})$ although $\theta_R^{[n]} \subset H^1(\omega^{[n]})$). Because the functions in $H^1(\omega^{[0]})$ with compact support are dense in $H_0^1(\omega^0)$, there is $\bar{w}^{[n]} \in H_0^1(\omega^{[n]})$ such that $\|w^{[0]} - \bar{w}^{[n]}\| \leqq \varepsilon$ for all $n \geqq n_2(\varepsilon)$. Hence with $\bar{\underline{u}}^{[n]} = (\bar{w}^{[n]}, \underline{\theta}_R^{[n]})$ we get

$$\|\bar{\underline{u}}^{[n]} - \underline{u}_R^{[0]}\| \to 0 \quad \text{as } n \to \infty.$$

Hence, using Theorem B.2 (Appendix B), we have

$$\|w_R^{[n]} - w_R^{[0]}\|_{H^1(\omega^{[n]})}^2 + \|\underline{\theta}^{[n]} - \underline{\theta}_R^{[0]}\|_{H^1(\omega^{[n]})}^2 \to 0 \quad \text{as } n \to \infty.$$

In summary, we have proved Theorem 4.4 below.

THEOREM 4.4. *Let $f \in L_2(\omega^{[0]})$ and let $u_R^{[n]} = (w_R^{[n]}, \underline{\theta}_R^{[n]})$, respectively, $u_R^{[0]} = (w_R^{[0]}, \underline{\theta}_R^{[0]})$, be the Reissner–Mindlin solution on $\omega^{[n]}$, respectively, $\omega^{[0]}$, for soft simple support and fixed h. Then*

$$\|w_R^{[n]} - w_R^{[0]}\|_{H^1(\omega^{[n]})} + \|\underline{\theta}_R^{[n]} - \underline{\theta}_R^{[0]}\|_{H^1(\omega^{[n]})} \to 0 \quad \text{as } n \to \infty.$$

We see that in contrast to the hard support there is no plate paradox when the soft support is imposed. Hence soft simple support is physically more natural than hard simple support.

Remark 4.3. In Theorem 4.2 we assumed that $f \in L_2(\omega^{[0]})$ while the solutions $\underline{u}_R^{[0]}$ and $\underline{u}_R^{[n]}$ were defined for any $f \in H^{-1}(\omega^{[0]})$, respectively, $f \in H^{-1}(\omega^{[n]})$. If f has compact support, then Theorem 4.4 also holds for $f \in H^{-1}(\omega^{[0]})$. We can weaken the assumptions on f in Theorem 4.4, e.g., so that $f \in H^{\alpha}(\omega^{[0]})$, $\alpha > -\frac{1}{2}$, but the proof will not hold for $f \in H^{-1}(\omega^{[0]})$.

Remark 4.4. We have assumed that $\omega^{[n]}$ is the sequence of regular polygons. This assumption was used only when we were using Theorem B.3. Hence Theorem 4.4 holds for any regular family of domains (see Appendix B). If f satisfies (4.6), then there is no need for regularity (see Remark 4.2) of the family of domains under consideration and Theorem 4.4 holds in the full generality.

Remark 4.5. We have assumed in Theorem 4.5 that $h > 0$ is fixed (i.e., independent of n). We could also consider a two-parameter family of problems where both n and h vary. Then, for n fixed and $h \to 0$, $\underline{u}_R^{[n]} \to \underline{u}_K^{[n]}$ (and hence for $h \to 0$ the difference between soft and hard support disappears). Hence, combining the results of this section with § 4.2, we see that

$$\lim_{n\to\infty} \lim_{h\to 0} \underline{u}_R^{(n)} \neq \lim_{h\to 0} \lim_{n\to\infty} \underline{u}_R^{(n)}.$$

In a way analogous to the proof of Theorem 4.4, we can prove Theorem 4.5.

THEOREM 4.5. *Let h be fixed and $\underline{u}^{[0]}$, respectively $\underline{u}^{[n]}$, be the solution of the three-dimensional plate problem on $\Omega^{[0]}$, respectively, $\Omega^{[n]}$, with soft simple support. Assume that the load $p \in L_2(\omega^{[0]})$. Then*

$$\|\underline{u}^{[0]} - \underline{u}^{[n]}\|_{1,\Omega^{[n]}} \to 0$$

as $n \to \infty$.

Remark 4.6. Remarks 4.3–4.5 are also valid for the three-dimensional plate model.

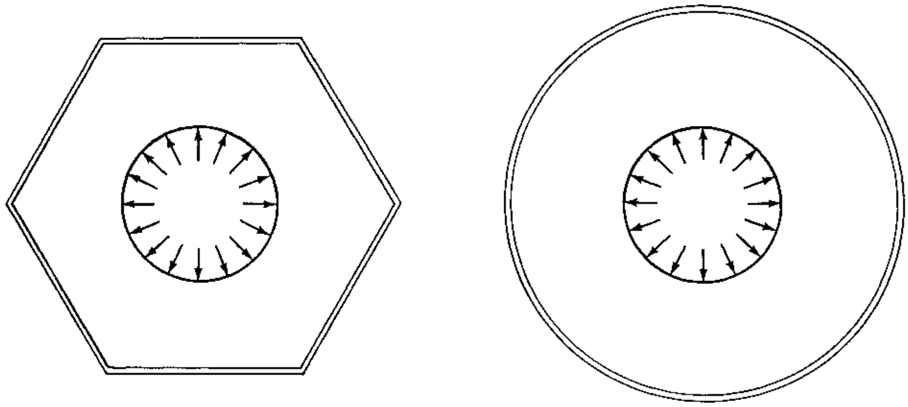
4.4. Some additional considerations. As we have seen, the Kirchhoff model (biharmonic equation) leads to paradoxical behavior for hard simple support. The same mathematical formulation also describes other problems and hence leads to the same paradoxical behavior.

As an example, we mention the problem of a reinforced tube shown in Fig. 4.3a, b. The reinforcement is attached by an unextendable tape to the exterior surface. Here we have the paradox that the stress caused by hydrostatic pressure is different for the polygonal and circular outer surfaces.

Analogous examples can very likely be found in fields other than elasticity where the problem reduces to the biharmonic (or polyharmonic) equation.

We have shown the paradoxical behavior for $n \to \infty$ and h relatively large compared with $1/n$ (see Remark 4.5). Hence the question arises of how large will be the difference between hard and soft support in three-dimensional formulation for n fixed and $h \to 0$. To this end we consider a square plate with sidelength equal to 1. In Table 4.1 we give the values of

$$\left[\frac{|E_{\text{SOFT}} - E_{\text{HARD}}|}{|E_{\text{SOFT}}|}\right]^{1/2} = \eta(h), \qquad \left[\frac{|E_{\text{HARD}} - E_K|}{|E_{\text{HARD}}|}\right]^{1/2} = \xi(h).$$



(a) (b)

FIG. 4.3. *Reinforced polygonal and circular tubes.*

TABLE 4.1

Percent	$h=0.1$	$h=0.01$
η	34.68	11.69
ξ	20.21	2.03

Here by E_{SOFT} and E_{HARD} we denote the (three-dimensional) plate energy for soft and hard support, and by E_K, the plate energy of the Kirchhoff model for the Poisson ratio $\nu=0$ (see also [7]).

Appendix A. Well-posedness of variational problems (3.4), (3.7), and (3.8). We use the following basic theorem (see [18]).

THEOREM A.1. *Let H be a Hilbert space and $\mathcal{B}$ be a bilinear form on $H\times H$ that satisfies*

$$\mathcal{B}(u,v)=\mathcal{B}(v,u),\qquad u,v\in H, \tag{A0}$$

$$|\mathcal{B}(u,v)|\leqq C\|u\|_H\|v\|_H,\qquad u,v\in H, \tag{A1}$$

$$\sup_{\substack{v\in H\\ \|v\|_H=1}} \mathcal{B}(u,v)\geqq c\|u\|_H\quad \forall u\in H, \tag{A2}$$

where C and c are positive constants. Then if F is any bounded linear functional on H, there is a unique $u\in H$ satisfying

$$\mathcal{B}(u,v)=F(v),\qquad v\in H. \tag{A3}$$

In applying Theorem A.1 to problems (3.4), (3.7), and (3.8), we choose the following notation.

(a) The three-dimensional model (3.4):

$$H=U\times\mathcal{H},$$

$$\mathcal{B}(\underline{u},\underline{\underline{\sigma}};\underline{v},\underline{\underline{\tau}})=(\underline{\underline{\sigma}}S^{-1}\underline{\underline{\tau}})_{\mathcal{H}}-(\underline{\underline{\varepsilon}}(\underline{u}),\underline{\underline{\tau}})_{\mathcal{H}}-(\underline{\underline{\sigma}},\underline{\underline{\varepsilon}}(\underline{v}))_{\mathcal{H}},$$

$$F(\underline{v},\underline{\underline{\tau}})=-Q(\underline{v}).$$

(b) The Reissner–Mindlin model (3.7):

$$H = H_0^1(\omega)\times V\times \mathcal{H}\times[L_2(\omega)]^2,$$

$$\mathcal{B}(w, \underline{\theta}, \underline{\underline{m}}, \underline{\gamma}; z, \underline{\varphi}, \underline{\underline{k}}, \underline{\zeta}) = (\underline{\underline{m}}, T^{-1}\underline{\underline{k}})_{\mathcal{H}} - (\underline{\underline{\varepsilon}}(\underline{\theta}), \underline{\underline{k}})_{\mathcal{H}} - (\underline{\underline{\varepsilon}}(\underline{\varphi}), \underline{\underline{m}})_{\mathcal{H}} - (\underline{\theta} - \underline{\nabla} w, \underline{\zeta}) - (\underline{\varphi} - \underline{\nabla} z, \underline{\gamma}) + (h^2/\kappa)(\underline{\gamma}, \underline{\zeta}),$$

$$F(z, \underline{\varphi}, \underline{\underline{k}}, \underline{\zeta}) = -\langle f, z\rangle.$$

(c) The Kirchhoff model (3.8):

$$H = W\times V\times \mathcal{H}\times V',$$

$$\mathcal{B}(w, \underline{\theta}, \underline{\underline{m}}, \underline{\gamma}; z, \underline{\varphi}, \underline{\underline{k}}, \underline{\zeta}) = (\underline{\underline{m}}, T^{-1}\underline{\underline{k}})_{\mathcal{H}} - (\underline{\underline{\varepsilon}}(\underline{\theta}), \underline{\underline{k}})_{\mathcal{H}} - (\underline{\underline{\varepsilon}}(\underline{\varphi}), \underline{\underline{m}})_{\mathcal{H}} - \langle \underline{\theta} - \underline{\nabla} w, \underline{\zeta}\rangle - \langle \underline{\varphi} - \underline{\nabla} z, \underline{\gamma}\rangle,$$

$$F(z, \underline{\varphi}, \underline{\underline{k}}, \underline{\zeta}) = -\langle f, z\rangle.$$

Then in each case, $\mathcal{B}$ is symmetric, F is a bounded linear functional on H, and the variational problem takes the general form (A3). Thus it suffices to show that (A1) and (A2) hold.

THEOREM A.2. *Assume that ω is a bounded Lipschitz domain and that the parameters ν, h, and κ satisfy*

$$0 \leqq \nu < \tfrac{1}{2}, \quad \bar{h} \leqq h \leqq \bar{h}^{-1}, \quad \bar{\kappa} \leqq \kappa \leqq \bar{\kappa}^{-1},$$

where $\bar{h} > 0$ and $\bar{\kappa} > 0$ are given. Then in each of the three cases above there are constants $C = C(\bar{h}, \bar{\kappa})$ and $c = c(\omega, \bar{h}, \bar{\kappa})$ such that (A1) *and* (A2) *hold.*

Proof. In view of (3.2) and (3.6) the mappings $S^{-1}: \mathcal{H}\to\mathcal{H}$ and $T^{-1}: \mathcal{H}\to\mathcal{H}$ are uniformly bounded in the assumed range of ν. It then follows easily that the assertion concerning (A1) holds, so let us concentrate on showing that (A2) is true.

(a) The three-dimensional model. Let $(\underline{u}, \underline{\underline{\sigma}}) \in U\times\mathcal{H}$ be given and let

$$(\underline{\underline{\sigma}}_0)_{ij} = \tfrac{1}{3}\operatorname{tr}(\underline{\underline{\sigma}})\delta_{ij}, \qquad i, j = 1, 2, 3.$$

Then $\|\underline{\underline{\sigma}}\|^2_{\mathcal{H}} = \|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_0\|^2_{\mathcal{H}} + \frac{1}{3}\|\operatorname{tr}(\underline{\underline{\sigma}})\|^2_{0,\Omega}$ and it follows from (3.2) that

$$(\text{A4}) \qquad \begin{aligned}(\underline{\underline{\sigma}}, S^{-1}\underline{\underline{\sigma}})_{\mathcal{H}} &= \frac{D}{E}\{(1+\nu)\|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_0\|^2_{\mathcal{H}} + (1-2\nu)\|\underline{\underline{\sigma}}_0\|^2_{\mathcal{H}}\}\\ &\geqq \frac{h^3}{12}\|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_0\|^2_{\mathcal{H}} \qquad (0 \leqq \nu \leqq \tfrac{1}{2}).\end{aligned}$$

We use the following lemma, which is related to the well-posedness of the Stokes problem. For the proof see [12].

LEMMA A.1. *There exists $\underline{v}_0 \in U$ and a constant C_1 depending on ω and $\bar{h}$ such that the following inequalities hold:*

$$\|\underline{v}_0\|_{1,\Omega} \leqq C_1\|\operatorname{tr}(\underline{\underline{\sigma}})\|_{0,\Omega},$$

$$(\operatorname{div}\underline{v}_0, \operatorname{tr}(\underline{\underline{\sigma}})) \geqq \|\operatorname{tr}(\underline{\underline{\sigma}})\|^2_{0,\Omega}.$$

With $\underline{v}_0$ as in Lemma A.1 we now set $(\underline{v}, \underline{\underline{\tau}}) = (-\underline{u} - \delta\underline{v}_0, \underline{\underline{\sigma}} - \delta^2\underline{\underline{\varepsilon}}(\underline{u}))$, where δ is a constant to be specified shortly. Then by (A4), the inequality $(\tau_1, \tau_2)_{\mathcal{H}} \leqq (s/2)\|\tau_1\|^2_{\mathcal{H}} + (1/2s)\|\tau_2\|^2_{\mathcal{H}}$ $(s > 0)$, and Lemma 3.1, we have that

$$\begin{aligned}\mathcal{B}(\underline{u}, \underline{\underline{\sigma}}; \underline{v}, \underline{\underline{\tau}}) &= (\underline{\underline{\sigma}}, S^{-1}\underline{\underline{\sigma}})_{\mathcal{H}} + \tfrac{1}{3}\delta(\operatorname{tr}(\underline{\underline{\sigma}}), \operatorname{div}\underline{v}_0)\\ &\quad + \delta(\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_0, \underline{\underline{\varepsilon}}(\underline{v}_0)) + \delta^2\|\underline{\underline{\varepsilon}}(\underline{u})\|^2_{\mathcal{H}} - \delta^2(\underline{\underline{\sigma}}, S^{-1}\underline{\underline{\varepsilon}}(\underline{u}))_{\mathcal{H}}\\ &\geqq (\tfrac{1}{12}\bar{h}^3 - C_2\delta - C_3\delta^2)\|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}_0\|^2_{\mathcal{H}} + (\tfrac{1}{6}\delta - C_4\delta^2)\|\operatorname{tr}(\underline{\underline{\sigma}})\|^2_{0,\Omega} + c_1\delta^2\|\underline{u}\|^2_{1,\Omega}\\ &\geqq \min\{\tfrac{1}{12}\bar{h}^3 - C_2\delta - C_3\delta^2, \tfrac{1}{2}\delta - 3C_4\delta^2, c_1\delta^2\}(\|\underline{u}\|^2_{1,\Omega} + \|\underline{\underline{\sigma}}\|^2_{\mathcal{H}}).\end{aligned}$$

Thus, choosing δ to be a sufficiently small positive number, we have found $(\underline{v}, \underline{\underline{\tau}}) \in U \times \mathcal{H}$ such that $\|\underline{v}, \underline{\underline{\tau}}\|_H \leqq C\|\underline{u}, \underline{\underline{\sigma}}\|_H$ and $\mathcal{B}(\underline{u}, \underline{\underline{\sigma}}; \underline{v}, \underline{\underline{\tau}}) \geqq c\|\underline{u}, \underline{\underline{\sigma}}\|_H^2$, where C and c depend only on ω and $\bar{h}$. Hence (A2) is true in case (a) with c depending on ω and $\bar{h}$.

(b) The Reissner–Mindlin model. Given $(w, \underline{\theta}, \underline{\underline{m}}, \underline{\gamma}) \in H_0^1(\omega) \times V \times \mathcal{K} \times [L_2(\omega)]^2$, let $(z, \underline{\varphi}, \underline{\underline{k}}, \underline{\zeta}) = (-w, -\underline{\theta}, \underline{\underline{m}} - \delta \underline{\underline{\varepsilon}}(\underline{\theta}), \underline{\gamma} - \delta(\underline{\theta} - \underline{\nabla} w))$, where δ is a constant to be specified. Then noting that by (3.6), $(\underline{\underline{m}}, T^{-1}\underline{\underline{m}}) \geqq \|\underline{\underline{m}}\|_{\mathcal{K}}^2/(1+\nu)$, and recalling Lemma 3.3, we have

$$\begin{aligned}\mathcal{B}(w, \underline{\theta}, \underline{\underline{m}}, \underline{\gamma}; z, \underline{\varphi}, \underline{\underline{k}}, \underline{\zeta}) &= (\underline{\underline{m}}, T^{-1}\underline{\underline{m}})_{\mathcal{K}} + \left(\frac{h^2}{\kappa}\right)\|\underline{\gamma}\|_{0,\omega}^2 + \delta\|\underline{\underline{\varepsilon}}(\underline{\theta})\|_{\mathcal{K}}^2 - \delta(\underline{\underline{m}}, T^{-1}\underline{\underline{\varepsilon}}(\underline{\theta}))_{\mathcal{K}} \\ &\quad + \delta\|\underline{\theta} - \underline{\nabla} w\|_{0,\omega}^2 - \delta\left(\frac{h^2}{\kappa}\right)(\underline{\gamma}, \underline{\theta} - \underline{\nabla} w) \\ &\geqq \left(\frac{1}{1+\nu} - C_1\delta\right)\|\underline{\underline{m}}\|_{\mathcal{K}}^2 + \frac{1}{2}\delta\|\underline{\underline{\varepsilon}}(\underline{\theta})\|_{\mathcal{K}}^2 + \frac{1}{2}\delta\|\underline{\theta} - \underline{\nabla} w\|_{0,\omega}^2 \\ &\quad + \left(\frac{h^2}{\kappa}\right)\left(1 - C_2\delta\frac{h^2}{\kappa}\right)\|\underline{\gamma}\|_{0,\omega}^2 \\ &\geqq \min\left\{\frac{1}{1+\nu} - C_1\delta, c_1\delta, \frac{h^2}{\kappa}\left(1 - C_2\delta\frac{h^2}{\kappa}\right)\right\}\|w, \underline{\theta}, \underline{\underline{m}}, \underline{\gamma}\|_H^2.\end{aligned}$$

Thus if δ is small enough we have found $(z, \underline{\varphi}, \underline{\underline{k}}, \underline{\zeta}) \in H$ such that $\|z, \underline{\varphi}, \underline{\underline{k}}, \underline{\zeta}\|_H \leqq C\|w, \underline{\theta}, \underline{\underline{m}}, \underline{\gamma}\|_H$ and $\mathcal{B}(w, \underline{\theta}, \underline{\underline{m}}; z, \underline{\varphi}, \underline{\underline{k}}, \underline{\zeta}) \geqq c\|w, \underline{\theta}, \underline{\underline{m}}, \underline{\gamma}\|_H^2$, where the constants depend only on ω, $\bar{h}$, and $\bar{\kappa}$. These prove the assertion in case (b).

(c) The Kirchhoff model. Given $(w, \underline{\theta}, \underline{\underline{m}}, \underline{\gamma}) \in H$, let $(z, \underline{\varphi}, \underline{\underline{k}}, \underline{\zeta}) = (-w, -\underline{\theta} - \delta\underline{\varphi}_0, \underline{\underline{m}} - \delta\underline{\underline{\varepsilon}}(\underline{\theta}), \underline{\gamma} - \delta\underline{\zeta}_0)$, where $\underline{\varphi}_0 \in V$ and $\underline{\zeta}_0 \in V'$ are defined so as to satisfy

$$\|\underline{\varphi}_0\|_{1,\omega} = \|\underline{\gamma}\|_{V'}, \qquad \langle \underline{\gamma}, \underline{\varphi}_0 \rangle = \|\underline{\gamma}\|_{V'}^2,$$
$$\|\underline{\zeta}_0\|_{V'} = \|\underline{\theta} - \underline{\nabla} w\|_{1,\omega}, \qquad \langle \underline{\theta} - \underline{\nabla} w, \underline{\zeta}_0 \rangle = \|\underline{\theta} - \underline{\nabla} w\|_{1,\omega}^2,$$

which obviously is possible. As in case (b), we then find that for a sufficiently small δ, $\|z, \underline{\varphi}, \underline{\underline{k}}, \underline{\zeta}\|_H \leqq C\|w, \underline{\theta}, \underline{\underline{m}}, \underline{\gamma}\|_H$ and $\mathcal{B}(w, \underline{\theta}, \underline{\underline{m}}, \underline{\gamma}; z, \underline{\varphi}, \underline{\underline{k}}, \underline{\zeta}) \geqq c\|w, \underline{\theta}, \underline{\underline{m}}, \underline{\gamma}\|_H^2$, where C and c depend only on ω, and so the assertion follows in case (c).

Appendix B. The Korn inequality. Let ω be a bounded Lipschitz domain and define the seminorm

$$|\underline{\theta}|_{E(\omega)} = \left\{\int_\omega \sum_{i,j=1}^{2} |\varepsilon_{ij}(\underline{\theta})|^2 \, dx_1 \, dx_2\right\}^{1/2}, \qquad \underline{\theta} \in (H^1(\omega))^2,$$

where $\varepsilon_{ij}(\underline{\theta}) = \frac{1}{2}(\partial\theta_i/\partial x_j + \partial\theta_j/\partial x_i)$, and let

$$|\underline{u}|_{R,\omega}^2 = |\underline{\theta}|_{E(\omega)}^2 + \|\underline{\theta} - \underline{\nabla} w\|_{0,\omega}^2, \quad \underline{u} = (w, \underline{\theta}), \quad w \in H^1(\omega), \quad \underline{\theta} \in (H^1(\omega))^2.$$

THEOREM B.1. *There is a constant C depending only on ω such that for any $\underline{\theta} \in [H^1(\omega)]^2$*

$$\text{(B1)} \qquad \inf_{abc}\{\|\theta_1 - a - bx_2\|_{1,\omega}^2 + \|\theta_2 - c + bx_1\|_{1,\omega}^2\} \leqq C|\underline{\theta}|_{E(\omega)}^2,$$

$$\text{(B2)} \qquad \inf_{abcd} \|w - (a + bx_1 + cx_2 + dx_1x_2)\|_{H^1(\omega)} \leqq C|\underline{u}|_{R,\omega}.$$

Proof. Inequality (B1) follows immediately from the Korn inequality for plane elasticity (see [19]). Inequality (B2) follows from (B1).

LEMMA B.1. *There exists a constant C depending on ω such that for any $(w, \underline{\theta}) \in [H^1(\omega)]^3$*

$$\|w\|_{1,\omega}^2 + \|\underline{\theta}\|_{1,\omega}^2 \leqq C\left\{|\underline{u}|_{R,\omega}^2 + \int_{\partial\omega} w^2\, ds\right\}.$$

Proof. We apply the standard contradiction argument. If the assertion is not true, there is a sequence $\{w_n, \underline{\theta}_n\}$ such that

$$\|w_n, \underline{\theta}_n\|_{1,\omega} = 1,$$

$$\|\underline{\theta}_n\|_{E(\omega)} \to 0,$$

$$\|\underline{\theta}_n - \underline{\nabla} w_n\|_{0,\omega} \to 0, \qquad \int_{\partial\omega} w_n^2\, ds \to 0$$

as $n \to \infty$. Then by Theorem B.1, $\{\underline{\theta}_n\}$ contains a subsequence (which we denote once more by $\{\underline{\theta}_n\}$) such that $\underline{\theta}_n \to (a - bx_2, c + bx_1)$ in $[H^1(\omega)]^2$. Furthermore, since $\|\theta_n - \nabla w_n\|_{0,\omega} \to 0$, there is another subsequence (once more denoted by $\{\underline{\theta}_n, w_n\}$) so that $w_n \to w$ in $H^1(\omega)$. Hence $b = 0$ and $w = ax_1 + cx_2 + d$. Because $\int w_n^2\, ds \to 0$ we get $a = c = d = 0$, contradicting the assumption $\|w_n, \underline{\theta}_n\|_{1,\omega} = 1$. □

We immediately get Theorem B.2.

THEOREM B.2. *There exists a constant C depending only on ω such that for any $\underline{u} = (w, \underline{\theta}) \in H_0^1(\omega) \times [H^1(\omega)]^2$*

$$\text{(B3)} \qquad \|w\|_{1,\omega}^2 + \|\underline{\theta}\|_{1,\omega} \leqq C|\underline{u}|_{R,\omega}^2.$$

Let us now consider a family $\mathscr{F} = \{\omega\}$ of Lipschitz bounded domains. The family will be called *regular* if there is a (uniform) constant C so that (B3) holds for all $\omega \in \mathscr{F}$.

Let us now consider a special family of domains. Let $\omega^{[0]}$ be a unit circle and $\omega^{[n]}$ be a sequence of regular $n+3$-polygons such that

$$\bar{\omega}^{[n]} \subset \bar{\omega}^{[n+1]} \subset \bar{\omega}^{[n+1]} \subset \omega^{[0]},$$

$$\omega^{[n]} \to \omega^{[0]} \quad \text{as } n \to \infty$$

in the sense that for any $x \in \omega^{[0]}$ there is $n(x) > 0$ such that $x \in \omega^{[n]}$ for all $n > n(x)$. We let $\mathscr{F}_0 = \{\omega^{[0]}, \omega^{[1]}, \omega^{[2]}, \cdots\}$.

THEOREM B.3. *The family $\mathscr{F}_0$ is a regular family of domains and hence there exists $C > 0$ such that*

$$\|w\|_{1,\omega^{[n]}}^2 + \|\underline{\theta}\|_{1,\omega^{[n]}}^2 \leqq C|\underline{u}|_{R,\omega^{[n]}}^2$$

for any $\underline{u} = (w, \underline{\theta}) \in H_0^1(\omega^{[n]}) \times [H^1(\omega^{[n]})]^2$, $n = 0, 1, 2, \cdots$.

Proof. For $n > n_0$ the $\omega^{[n]}$ are star-shaped domains and

$$\partial\omega^{[n]} = \{(x_1, x_2): x_1 = \rho_n(\theta)\cos\theta, x_2 = \rho_n(\theta)\sin\theta, 0 \leqq \theta \leqq 2\pi\},$$

where $\rho_n(\theta) \to 1$ and $\rho_n'(\theta) \to 0$ uniformly. Let Q_n be the one-to-one map of $\omega^{[n]}$ onto $\omega^{[0]}$ defined by

$$Q_n(\rho(\theta)\cos\theta, \rho(\theta)\sin\theta) = (\rho(\theta)\cos\theta, \rho(\theta)\sin\theta) \quad \text{for } \rho(\theta) \leqq \frac{1}{2}$$

$$= \left(\frac{1}{2}\frac{\rho(\theta) - (\frac{1}{2})}{\rho_n(\theta) - (\frac{1}{2})} + \frac{1}{2}\right)\cos\theta$$

$$= \left(\frac{1}{2}\frac{\rho(\theta) - (\frac{1}{2})}{\rho_n(\theta) - (\frac{1}{2})} + \frac{1}{2}\right)\sin\theta \quad \text{for } \rho(\theta) > \frac{1}{2}.$$

If $Q_n(x_1, x_2) = (\xi_1, \xi_2)$ then we have $\xi_1 = \xi_1^{[n]}(x_1, x_2)$, $\xi_2^{[n]} = \xi_2(x_1, x_2)$, $x_1 = x_1^{[n]}(\xi_1, \xi_2)$, $x_2 = x_2^{[n]}(\xi_1, \xi_2)$, and $\xi_i^{[n]} \to x_i$, $(\partial \xi_i/\partial x_j) \to \delta_{ij}$, $x_i^{[n]} \to \xi_i$, $(\partial x_i/\partial \xi_j) \to \delta_{ij}$, $i, j = 1, 2$ as $n \to \infty$, uniformly with respect to $(x_1, x_2) \in \omega^{[n]}$ and $(\xi_1, \xi_2) \in \omega^{[0]}$. Let $\underline{u} = (w, \underline{\theta}) \in H_0^1(\omega^{[n]}) \times (H^1(\omega^{[n]}))^2$ and let

$$\bar{u} = (\bar{w}, \bar{\underline{\theta}}), \qquad \bar{\underline{u}}(\xi_1, \xi_2) = \underline{u}(x_1(\xi_1, \xi_2), x_2(\xi_1, \xi_2)).$$

Then $\bar{\underline{u}} \in H_0^1(\omega^{[0]}) \times (H^1(\omega^{[0]})^2)$ and by Theorem B.2 we have

$$\|\bar{w}\|_{1,\omega^{[0]}}^2 + \|\underline{\theta}\|_{1,\omega^{[0]}}^2 \leqq C|\bar{\underline{u}}|_{R,\omega^{[0]}}^2$$

and also

$$\|\bar{w}\|_{1,\omega^{[0]}} = \|w\|_{1,\omega^{[n]}}(1 + o(1)),$$

$$\|\bar{\underline{\theta}}\|_{1,\omega^{[0]}} = \|\underline{\theta}\|_{1,\omega^{[n]}}(1 + o(1)),$$

$$|\bar{\underline{u}}|_{R,\omega^{[0]}} = |\underline{u}|_{R,\omega^{[n]}} + o(1)(\|w\|_{1,\omega^{[n]}} + \|\underline{\theta}\|_{1,\omega^{[n]}})$$

as $n \to \infty$. Hence

$$\|w\|_{1,\omega^{[n]}}^2(1 + o(1)) + \|\underline{\theta}\|_{1,\omega^{[n]}}^2(1 + o(1)) \leqq C[|\underline{u}|_{R,\omega^{[n]}}^2 + o(1)(\|w\|_{1,\omega^{[n]}}^2 + \|\underline{u}\|_{1,\omega^{[n]}}^2)].$$

From this we see that for $n > n_0$ the family is a regular one. Using Theorem B.2, we then see that the whole family $\mathscr{F}_0$ is regular. □

Appendix C. A projection theorem.

THEOREM C.1. *Let H be a Hilbert space, let $\{H_n\}$ and $\{K_n\}$ be sequences of closed subspaces of H such that $H_n \subset H_{n+1}$ and $K_n \supseteq K_{n+1}$, $n = 1, 2, \cdots$, and let*

$$H_0 = \overline{\bigcup_n H_n} \quad \text{and} \quad K_0 = \bigcap_n K_n.$$

Furthermore, let P_n and Q_n, respectively, P_0, Q_0, be orthogonal projections onto H_n and K_n, respectively, H_0, K_0. Then for any $u \in H$

$$\|P_n u - P_0 u\| \to 0, \qquad \|Q_n u - Q_0 u\| \to 0$$

as $n \to \infty$.

Proof. First observe that $\|Q_{n+1}u\| = \|Q_{n+1}Q_n u\| \leqq \|Q_n u\|$, so $\|Q_n u\| \to q \geqq 0$ monotonically. Furthermore,

$$\|Q_n u - Q_{n+j}u\| = \|Q_n u\|^2 - 2(Q_n u, Q_{n+j}u) + \|Q_{n+j}\|^2 = \|Q_n u\|^2 - \|Q_{n+j}u\|^2,$$

so $\{Q_n u\}$ is a Cauchy sequence. So $Q_n u \to v$ and $v \in K_n$ for all n. Hence $v \in K_0$ and since $(v, w) = \lim_{n\to\infty}(Q_n u, w) = \lim_{n\to\infty}(u, Q_n w) = (u, w)$ for all $w \in K_0$, it follows that $v = Q_0 u$.

Let us now consider the projection operator $I - P_n = \tilde{Q}_n$. Then $\tilde{Q}_n$ projects H onto H_n^1 and $H_w^1 \supset H_{n+1}^1$. Hence $\tilde{Q}_n u = u - P_n u \to u - v \in \bigcap H_n^1$. So $P_n u \to v \in H_0$ and by the same argument as before, $v = P_0 u$. □

REFERENCES

[1] S. A. AMBARCUMJAN, *Theory of Anisotropic Plates*, Second edition, Nauka, Moscow, 1987. (In Russian.)

[2] I. BABUŠKA, *Stability of the domain under perturbation of the boundary in fundamental problems in the theory of partial differential equations principally in connection with the theory of elasticity* 1, 2, Czechoslovak Math. J., 11 (1961), pp. 75–105, 165–203. (In Russian.)

[3] ———, *The stability of domains and the question of the formulation of the plate problem*, Apl. Mat., (1962), pp. 463–467.

[4] I. BABUŠKA, *The theory of small changes in the domain of existence in the theory of partial differential equations and its applications*, in Differential Equations and Their Applications, Proc. Conference, Prague, 1962, Academic Press, New York, 1963, pp. 13-26.

[5] I. BABUŠKA AND A. K. AZIZ, *Survey lectures on the mathematical foundations of the finite element method*, in The Mathematical Foundations of the Finite Element Method with Application to Partial Differential Equations, A. K. Aziz, ed., Academic Press, New York, 1973, pp. 5-363.

[6] I. BABUŠKA, *Uncertainties in engineering design: mathematical theory and numerical experience*, in Optimal Shape—Automated Structural Design, J. Bennett and M. Botkin, eds., Plenum Press, New York, 1986.

[7] I. BABUŠKA AND T. SCAPOLLA, *Benchmark computation and performance evaluation for a rhombic plate bending problem*, Internat. J. Numer. Methods Engrg., 28 (1989), pp. 155-181.

[8] G. BIRKHOFF, *The Numerical Solution of Elliptic Equations*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1971.

[9] F. BREZZI, *On the existence, uniqueness and approximation of saddle point problems arising from Lagrange multipliers*, RAIRO, Ser R., 8 (1974), pp. 129-151.

[10] P. G. CIARLET AND P. DESTUYNDER, *A justification of the two-dimensional linear plate model*, J. Méc. Théor. Appl., 18 (1979), pp. 315-344.

[11] P. DESTUYNDER, *Une théorie asymptotique des plaques minces en élasticité linéaire*, Masson, Paris, New York, 1986.

[12] V. GIRAULT AND P. A. RAVIART, *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms*, Springer-Verlag, Berlin, New York, 1986.

[13] P. GRISVARD, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston, 1985.

[14] V. G. MAZ'YA AND S. A. NAZAROV, *About the Sapondzhyn-Babuška paradox in the plate theory*, Dokl. Akad. Nauk Armenian Rep., 78 (1984), pp. 127-130. (In Russian.)

[15] ———, *Paradoxes of limit passage in solutions of boundary value problems involving the approximation of smooth domains by polygonal domains*, Izv. Akad. Nauk SSSR Ser. Mat., 50 (1986), pp. 1156-1177. (In Russian.) Math. USSR-Izv., 29 (1987), pp. 511-533. (In English.)

[16] D. MORGENSTERN, *Herleitung der Plattentheorie aus der dreidimensionalen Elastizitätstheorie*, Arch. Rational Mech. Anal., 4 (1959), pp. 145-152.

[17] D. MORGENSTERN AND I. SZABÓ, *Vorlesungen über theoretische Mechanik*, Springer-Verlag, Berlin, New York, 1961.

[18] N. W. MURRAY, *The polygon-circle paradox in thin plate theory*, Proc. Cambridge Philos. Soc., 73 (1973), pp. 279-282.

[19] J. NEČAS AND I. HLAVACEK, *Mathematical Theory of Elastic and Elasto-Plastic Bodies: An Introduction*, Elsevier, Amsterdam, New York, 1981.

[20] J. PITKÄRANTA, *Analysis of some low-order finite element schemes for Mindlin-Reissner and Kirchhoff plates*, Numer. Math., 53 (1988), pp. 237-254.

[21] K. RAJAIAH AND A. K. RAO, *On the polygon-circle paradox*, Trans. ASME Ser. E., J. Appl. Mech., 48 (1981), pp. 195-196.

[22] E. REISSNER, *Reflections on the theory of elastic plates, developments in mechanics*, Appl. Mech. Rev., 38 (1985), pp. 1453-1464.

[23] G. RIEDER, *On the plate paradox of Sapondzhyan and Babuška*, Mech. Res. Comm., 1 (1974), pp. 51-53.

[24] A. S. SAADA, *Elasticity Theory and Applications*, Pergamon Press, New York, 1974.

[25] O. M. SAPONDZHYAN, *Bending of a freely supported polygonal plate*, Izv. Akad. Nauk Armyan SSR Ser. Fiz-Mat. Estestv. Tekhn. Nauk, 5 (1952), pp. 29-46.