SIAM J. NUMER. ANAL.
Vol. 31, No. 4, pp. 945–981, August 1994

001

SPECIAL FINITE ELEMENT METHODS FOR A CLASS OF SECOND ORDER ELLIPTIC PROBLEMS WITH ROUGH COEFFICIENTS*

IVO BABUŠKA†, GABRIEL CALOZ‡, AND JOHN E. OSBORN§

Abstract. In this paper the approximate solution of a class of second order elliptic equations with rough coefficients is considered. Problems of the type considered arise in the analysis of unidirectional composites, where the coefficients represent the properties of the material. Several methods for this class of problems are presented, and it is shown that they have the same accuracy as usual methods have for problems with smooth coefficients. The methods are referred to as special finite element methods because they are of finite element type but employ special shape functions, chosen to accurately model the unknown solution.

Key words. special finite elements, rough coefficients

AMS subject classification. 65N30

1. Introduction. In this paper we consider the approximate solution of a class of second order, two-dimensional elliptic boundary value problems with rough or highly oscillating coefficients. We apply an approach proposed by Babuška and Osborn [5] for the approximate solution of problems with rough input data. This approach was applied to one-dimensional problems in Babuška and Osborn [4]. Specifically, we consider boundary value problems of the form

$$\begin{cases} Lu(x,y) \equiv -\dfrac{\partial}{\partial x}\left(a(x,y)\dfrac{\partial}{\partial x}u(x,y)\right) - \dfrac{\partial}{\partial y}\left(a(x,y)\dfrac{\partial}{\partial y}u(x,y)\right) = f(x,y)\ \forall (x,y)\in\Omega, \\ u(x,y) = 0 \quad \forall (x,y) \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$, f is a function in $L^2(\Omega)$, and where the function $a \in L^\infty(\Omega)$ satisfies

$$0 < \alpha \le a(x,y) \le \beta < \infty \qquad \forall (x,y) \in \Omega, \tag{1.2}$$

where α and β are constants. Throughout most of the paper we also assume that $a(x,y)$ locally varies sharply in at most one direction, a requirement on the coefficient a that will be made precise later (see Remarks 2.1 and 4.1); such coefficients are also called (curvilinear or straight line) unidirectional. If the coefficient a is rough, then the solution u to (1.1) will also be rough; to be specific, u will not in general be in $H^2(\Omega)$ and may not be in $H^{1+\varepsilon}(\Omega)$ for any $\varepsilon > 0$.

Problems of this type arise in many applications; we are especially concerned with applications to unidirectional composite materials (briefly, composites). In these applications the coefficient $a(x,y)$ represents the properties of the material and changes

*Received by the editors March 10, 1992; accepted for publication (in revised form) July 27, 1993.

†Institute for Physical Science and Technology and Department of Mathematics, University of Maryland, College Park, Maryland 20742. The work of this author was supported by Office of Naval Research contract N00014-90-J-1030 and by National Science Foundation grant CCR-88-20279.

‡Institut de Recherche Mathématique de Rennes, Université de Rennes I, 35042 Rennes Cedex, France (caloz@lie.univ-rennes1.fr). The work of this author was partially supported by the Swiss National Science Foundation.

§Department of Mathematics and Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742 (jeo@math.umd.edu). The work of this author was partially supported by National Science Foundation grant DMS-84-10324.

abruptly. We are also interested in problems in which $a(x, y)$ changes smoothly but rapidly. We take the liberty of referring to both types of problems as composites. In Figs. 1.1–1.3 we show some typical configurations for unidirectional composites. In these figures the coefficient is constant or is changing slowly along the lines or curves and is changing sharply in the transverse direction; the

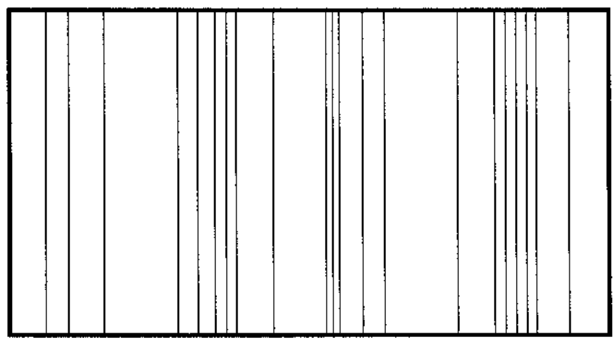


FIG. 1.1. *A straight line unidirectional composite.*

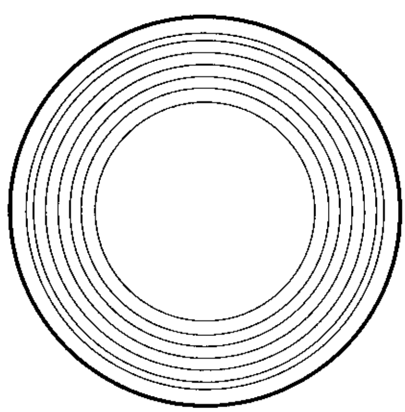


FIG. 1.2. *A tubular (curvilinear unidirectional) composite.*

absence of lines in a portion of the material indicates a constant or a slowly varying coefficient. We can interpret the lines as fibers in the composite. This interpretation is, of course, symbolic for problems in which $a(x, y)$ changes smoothly but rapidly.

Figure 1.1 shows a straight line unidirectional composite or coefficient, and Fig. 1.2 shows the cross-section of a tubular composite. Figs. 1.3a,b show reinforced panels. The area A in Fig. 1.3b indicates a region in which $a(x, y)$ is smoothly becoming a constant or a smooth function. We refer to the materials or the coefficients in Figs. 1.2–1.3 as curvilinear unidirectional. We note that certain interface problems can be naturally treated as problems of composites. With this approach it is not necessary to fit the interface with the finite elements, as is done with the standard approach.

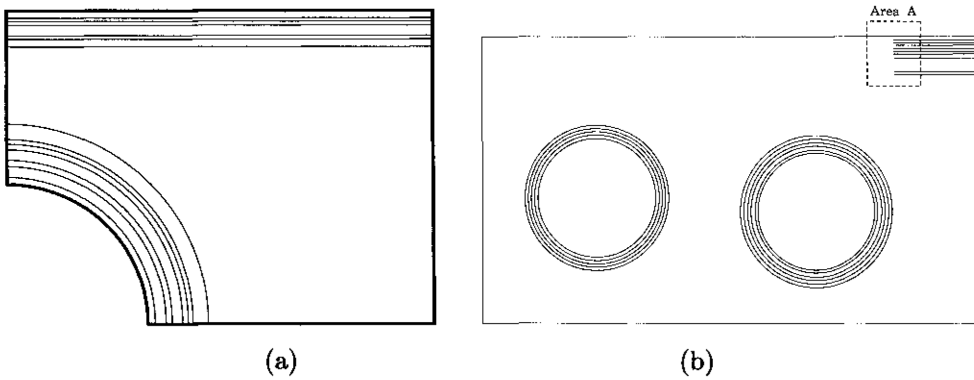


FIG. 1.3. *Reinforced panel.*

A finite element method is obtained by restricting the weak formulation of problem (1.1),

$$(1.3)\begin{cases} u \in H_0^1(\Omega), \\ B(u,v) \equiv \int_\Omega a \ \ \underset{\sim}{\mathrm{grad}} \ \ u \cdot \ \underset{\sim}{\mathrm{grad}} \ \ v\, dx\, dy = \int_\Omega fv\; dx\, dy \qquad \forall v \in H_0^1(\Omega), \end{cases}$$

to finite dimensional trial and test spaces. The outline of the approach given in [5] is as follows:

(1) Characterize the space of solutions corresponding to the space of right-hand sides (in our case we suppose $f \in L^2(\Omega)$). This will involve a regularity result. Although regularity results are well known for elliptic problems with smooth coefficients, they are not available in a direct form for our problem. Such results are discussed in §2.

(2) Select trial spaces that have good approximation properties. The approximation properties of the trial functions or shape functions are directly tied to the regularity of the solution. For example, if the solution u of (1.1) is not in $H^2(\Omega)$, then it is well known that the usual finite element method based on piecewise linear approximating functions produces inaccurate results. The problem of selecting optimal trial functions is not simple; in practice, one would like to find a trial space that performs almost as well as the optimal one but that can be reasonably implemented. We use the phrase special finite element methods to denote methods with this type of special shape functions.

(3) Select a test space so as to ensure the inf-sup (or stability) condition is satisfied and so that the resulting finite element method can be reasonably implemented.

We use this approach to design methods of finite element type that will yield, roughly speaking, the same accuracy as the usual finite element method when a is smooth, but strikingly improved accuracy when a is rough.

The organization of the paper is as follows. In §2 we present the regularity results needed for the problems we are dealing with. Then we propose and analyze several methods to solve problem (1.1) in the special case in which $\Omega = \Omega_0 = (0,1) \times (0,1)$ and $a(x,y) = a(x)$ is a function of x only. This study is carried out in §3, where we propose three distinct approximation methods. A function $a(x,y) = a(x)$ of x only is an example of a function that locally varies sharply in at most one direction; in fact, such an $a(x,y)$ globally varies sharply in at most one direction. The coefficient $a(x,y)$ can also be referred to as straight line unidirectional (see Fig. 1.1). In §4 we present a further development of two of the methods from §3 to treat problems of the type depicted in Figs. 1.2 and 1.3 with curvilinear unidirectional coefficients.

As noted above the approach presented in this paper is thoroughly studied in the one-dimensional case in [4]. Techniques similar to special elements were used in Ciarlet, Natterer, and Varga [8] and in Crouzeix and Thomas [9] to handle degenerate one-dimensional elliptic problems. We also mention the recent work of Moussaoui and Ziani [15], which deals with the same kind of problems with a method similar to our Method I, presented in §3.1. Finally we mention the papers [3], [16], [17], which are related to our approach.

Throughout the paper, we use the $L^2(\Omega)$-based Sobolev spaces $H^k(\Omega)$, consisting of functions with partial derivatives of order less than or equal to k in $L^2(\Omega)$. These spaces are equipped with the norms and seminorms

$$\|u\|_{k,\Omega}^2 = \int_\Omega \sum_{|\alpha|\le k} |D^\alpha u|^2,$$

$$|u|_{k,\Omega}^2 = \int_\Omega \sum_{|\alpha|=k} |D^\alpha u|^2.$$

We also use the spaces $H^k(\Omega)$ for fractional k. $H_0^1(\Omega)$ consists of those functions in $H^1(\Omega)$ that vanish on $\partial\Omega$. We also use the space $H^{-1}(\Omega) = [H_0^1(\Omega)]'$. Throughout, C denotes a generic constant. When we say there exists $C = C(\alpha,\beta)$, we mean that C depends on the coefficient $a(x,y)$ only through its upper and lower bounds α and β (cf. (1.2)).

2. Regularity results. It is clear that problem (1.1) has a unique (weak) solution in $H_0^1(\Omega)$; compare (1.3). This is an immediate consequence of the Lax–Milgram theorem. Furthermore,

$$\|u\|_{1,\Omega} \le C(\alpha)\|f\|_{0,\Omega}.$$

But if $a(x,y)$ is rough, then u may not be in $H^{1+\varepsilon}(\Omega)$ for any $\varepsilon > 0$, and therefore we cannot expect any reasonable rate of convergence for the usual finite element method. Nevertheless, as a consequence of the assumption that $a(x,y)$ is unidirectional, the solution u does satisfy a regularity property that can be employed in the derivation of an approximation method for (1.1) with a good rate of convergence, even though $a(x,y)$ is rough.

It is the purpose of this section to prove such regularity results, first for the model problem consisting of (1.1) with $\Omega = \Omega_0 = (0,1)\times(0,1)$ and the coefficient $a(x,y)$ satisfying $a(x,y) = a(x)$, that is, with a straight line unidirectional coefficient (cf. Fig. 1.1), and then for the more general problem with a curvilinear unidirectional coefficient (cf. Figs. 1.2–1.3). Our main tool is a theorem of Bernstein [6], [13, §3.17] for elliptic equations in nondivergence form, which we now state.

Consider the problem

$$\text{(2.1)}\qquad \begin{cases} -a_{11}(x,y)\dfrac{\partial^2 u}{\partial x^2} - 2a_{12}(x,y)\dfrac{\partial^2 u}{\partial x \partial y} - a_{22}(x,y)\dfrac{\partial^2 u}{\partial y^2} = f & \text{in } \Omega, \\ u = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded convex domain in $\mathbb{R}^2$ with a Lipschitz and piecewise C^2 boundary $\partial\Omega$ and where the functions $a_{ij} \in L^\infty(\Omega)$ satisfy

$$\nu \sum_{i=1}^{2} \xi_i^2 \le \sum_{i,j=1}^{2} a_{ij}(x,y)\xi_i\xi_j \le \mu \sum_{i=1}^{2} \xi_i^2 \qquad \forall (x,y) \in \Omega, \qquad \forall \underset{\sim}{\xi} \in \mathbb{R}^2, \tag{2.2}$$

with $a_{21} = a_{12}$, where ν and μ are positive constants. Note that the equation in (2.1) is in nondivergence form.

THEOREM 2.1 (BERNSTEIN). *For each $f \in L^2(\Omega)$, problem (2.1) has a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Furthermore, there is a constant $C(\nu,\mu)$, depending on ν and μ but independent of f, such that*

$$\|u\|_{2,\Omega} \le c(\nu,\mu)\|f\|_{0,\Omega}. \tag{2.3}$$

Our hypothesis on Ω is not identical to the one in [13]. To prove that (2.3) is still valid for such a domain, one can use the a priori estimates given in [11, §3.1].

The first application of Bernstein's Theorem will give a regularity result for problem (1.1) when $\Omega = \Omega_0$ and $a(x,y) = a(x)$. Corresponding to problem (1.1), with this assumption, we define the space

$$H^L(\Omega) = \left\{ u \in H^1(\Omega) : a(x)\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in H^1(\Omega) \right\} \tag{2.4}$$

with the norm

$$\|u\|_{L,\Omega}^2 = \|u\|_{1,\Omega}^2 + |u|_{L,\Omega}^2, \tag{2.5a}$$

where

$$|u|_{L,\Omega}^2 = \int_\Omega \left(a \left| \frac{\partial}{\partial x}\left(a \frac{\partial u}{\partial x} \right) \right|^2 + a \left| \frac{\partial^2 u}{\partial x \partial y} \right|^2 + \frac{1}{a} \left| \frac{\partial^2 u}{\partial y^2} \right|^2 \right) dx\,dy. \tag{2.5b}$$

THEOREM 2.2. *Suppose $\Omega = \Omega_0$ and $a(x,y) = a(x)$. Then for each $f \in L^2(\Omega)$ the solution u of (1.1) is in $H_0^1(\Omega) \cap H^L(\Omega)$. Furthermore, there is a constant $C = C(\alpha,\beta)$, depending on α and β but independent of f, such that*

$$\|u\|_{L,\Omega} \le C(\alpha,\beta)\|f\|_{0,\Omega}. \tag{2.6}$$

Proof. Let u be the unique solution to (1.1) in $H_0^1(\Omega)$. We introduce the change of variables or mapping

$$\tilde{x}(x) = \int_0^x \frac{ds}{a(s)}, \qquad \tilde{y}(y) = y \tag{2.7}$$

and the notation

$$\tilde{u}(\tilde{x}(x), \tilde{y}(y)) = u(x,y), \qquad (x,y) \in \Omega. \tag{2.8}$$

The mapping (2.7) maps the domain Ω onto $\tilde{\Omega} = (0, \int_0^1 \frac{ds}{a(s)}) \times (0,1)$. We see that $\tilde{u} \in H^2(\tilde{\Omega})$ if and only if $u \in H^1(\Omega), a\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in H^1(\Omega)$, which is equivalent to $u \in H^L(\Omega)$. We also note that the weak formulation (1.3) of (1.1) is transformed into

$$(2.9) \qquad \begin{cases} \tilde{u} \in H_0^1(\tilde{\Omega}), \\ \int_{\tilde{\Omega}} \left(\frac{\partial \tilde{u}}{\partial \tilde{x}} \frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{a}^2 \frac{\partial \tilde{u}}{\partial \tilde{y}} \frac{\partial \tilde{v}}{\partial \tilde{y}} \right) d\tilde{x}\, d\tilde{y} = \int_{\tilde{\Omega}} \tilde{f} \tilde{a} \tilde{v} d\tilde{x} d\tilde{y} \quad \forall \tilde{v} \in H_0^1(\tilde{\Omega}). \end{cases}$$

The system (2.9) is simply the variational formulation of

$$(2.10) \qquad \begin{cases} -\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} - \tilde{a}^2 \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} = \tilde{a}\tilde{f} \quad \text{in } \tilde{\Omega}, \\ \tilde{u} = 0 \quad \text{on } \partial\tilde{\Omega}. \end{cases}$$

Note that while the equation in (1.1) is in divergence form, the equation in (2.10) is in nondivergence (as well as in divergence) form. As a consequence of Theorem 2.1, (2.10) is uniquely solvable in $H_0^1(\tilde{\Omega}) \cap H^2(\tilde{\Omega})$ and

$$(2.11) \qquad \|\tilde{u}\|_{2,\tilde{\Omega}} \leq C(\alpha, \beta) \|\tilde{a}\tilde{f}\|_{0,\tilde{\Omega}}.$$

Because (2.10) is uniquely solvable in $H_0^1(\tilde{\Omega})$, we conclude that $\tilde{u}$, as defined in (2.8), which satisfies (2.9), coincides with the solution of (2.10) and hence lies in $H_0^1(\tilde{\Omega}) \cap H^2(\tilde{\Omega})$ and satisfies (2.11). Thus, $u \in H_0^1(\Omega) \cap H^L(\Omega)$, which is the first conclusion in the theorem. If we change variables in the estimate (2.11) to return to the original variables, we obtain

$$\begin{aligned} \|u\|_{L,\Omega}^2 &= \int_\Omega u^2 dx\, dy + \int_\Omega \left(\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 \right) dx\, dy \\ &\quad + \int_\Omega \left(a \left| \frac{\partial}{\partial x}\left(a \frac{\partial u}{\partial x}\right)\right|^2 + a \left|\frac{\partial^2 u}{\partial x \partial y}\right|^2 + \frac{1}{a}\left|\frac{\partial^2 u}{\partial y^2}\right|^2 \right) dx\, dy \\ &= \int_{\tilde{\Omega}} \tilde{u}^2 \tilde{a} d\tilde{x} d\tilde{y} + \int_{\tilde{\Omega}} \left(\frac{1}{\tilde{a}} \left|\frac{\partial \tilde{u}}{\partial \tilde{x}}\right|^2 + \tilde{a} \left|\frac{\partial \tilde{u}}{\partial \tilde{y}}\right|^2 \right) d\tilde{x}\, d\tilde{y} \\ &\quad + \int_{\tilde{\Omega}} \left(\left|\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}\right|^2 + \left|\frac{\partial^2 \tilde{u}}{\partial \tilde{x} \partial \tilde{y}}\right|^2 + \left|\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right|^2 \right) d\tilde{x}\, d\tilde{y} \\ &\leq \max\left(\beta, \frac{1}{\alpha}\right) \|\tilde{u}\|_{2,\tilde{\Omega}}^2 \\ &\leq \max\left(\beta, \frac{1}{\alpha}\right) C^2(\alpha, \beta) \|\tilde{a}\tilde{f}\|_{0,\tilde{\Omega}}^2 \\ &\leq \beta^2 \max\left(\beta, \frac{1}{\alpha}\right) C^2(\alpha, \beta) \|f\|_{0,\Omega}^2, \end{aligned}$$

which is (2.6). ☐

Theorem 2.2 was proved by making a global change of variables and then applying the Bernstein result. The global change of variables exists because $a(x, y)$ globally varies sharply in one direction: $a(x, y) = a(x)$. We now prove a second result in which we assume the existence of only a local change of variables (cf. Figs. 1.2–1.3).

Let $\Sigma \subset \Omega$ be open and assume that we have a system of orthogonal curvilinear coordinates (ξ, η) defined on Σ. More precisely, regarding Σ and (ξ, η) we assume

(i) the functions ξ, η are defined on $\overline{\Sigma}$ and are twice continuously differentiable,

(ii) $(\xi, \eta) : \Sigma \to \Sigma'$ is one-to-one and onto,

(iii) $\frac{\partial(\xi,\eta)}{\partial(x,y)} \geq \gamma > 0$ on Σ,

(iv) $\underset{\sim}{\text{grad}} \;\; \xi \cdot \;\; \underset{\sim}{\text{grad}} \;\; \eta = 0$ in Σ,

(v) Σ is a rectangle in ξ, η, that is, $\Sigma' = (\Sigma_\Sigma^1, \xi_\Sigma^2) \times (\eta_\Sigma^1, \eta_\Sigma^2)$, and

(vi) $\overline{\Sigma} \cap \partial\Omega = \begin{cases} \phi, \text{ in which case all edges of } \Sigma \text{ are called interior edges} \\ \text{or} \\ \text{the union of one or more edges of } \Sigma, \text{ in which case these edges are called boundary edges and the remaining edges are called interior edges.} \end{cases}$

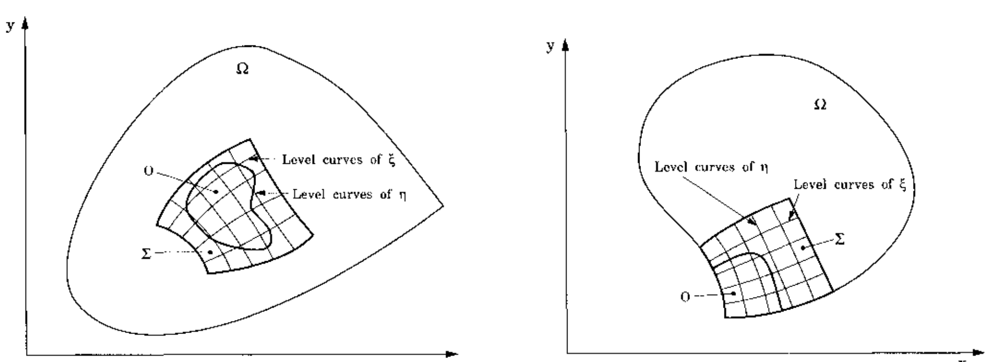
The union of the interior edges is denoted by E. We suppose further that

$$a(x,y) = a'(\xi) \qquad \forall (x,y) \in \Sigma, \tag{2.12}$$

where we use the notation, for any function w defined in Σ,

$$w'(\xi(x,y), \eta(x,y)) = w(x,y), \qquad (x,y) \in \Sigma.$$

See Figs. 2.1a,b for typical configurations.



(a) (b)

FIG. 2.1.

THEOREM 2.3. *Let u be the solution in $H_0^1(\Omega)$ of* (1.1), *where we assume that $f \in L^2(\Omega)$ and that $a(x,y)$ satisfies assumptions* (1.2) *and* (2.12), *where $\Sigma, (\xi, \eta)$ satisfies conditions* (i)–(vi) *above. Let $\mathcal{O} \subset \Sigma$ be open and satisfy $\mathcal{O} \subset\subset \Sigma$ if $\overline{\Sigma} \cap \partial\Omega = \phi$ and $\partial\mathcal{O} \cap \partial\Sigma \subset \partial\Omega$ if $\overline{\Sigma} \cap \partial\Omega \neq \phi$, and let $\mathcal{O}'$ be the image of $\mathcal{O}$ under the mapping (ξ, η) (cf. Figs.* 2.1a,b). *Then there is a constant $C = C(\alpha, \beta, \xi, \eta, d)$ depending on α, β, ξ, η, and d but independent of f, such that*

$$\left(\int_{\mathcal{O}'} \left[a' \left| \frac{\partial}{\partial \xi} \left(a' \frac{\partial u'}{\partial \xi} \right) \right|^2 + a' \left| \frac{\partial^2 u'}{\partial \xi \partial \eta} \right|^2 + \frac{1}{a'} \left| \frac{\partial^2 u'}{\partial \eta^2} \right|^2 \right] d\xi d\eta \right)^{1/2} \tag{2.13}$$
$$\leq C(\alpha, \beta, \xi, \eta, d) \|f\|_{0,\Omega},$$

where

$$d = \begin{cases} \text{dist}(\mathcal{O}, E) & \text{if } E \neq \phi, \\ 1, \text{if } E = \phi. \end{cases}$$

Proof. Clearly u (more precisely, $u|_\Sigma$) satisfies

$$(2.14) \qquad \begin{cases} u \in H^1(\Sigma), \\ \int_\Sigma a \ \ \underset{\sim}{\text{grad}} \ \ u \cdot \ \underset{\sim}{\text{grad}} \ \ v \ dx\, dy = \int_\Sigma fv \ dx\, dy \quad \forall v \in H_0^1(\Sigma). \end{cases}$$

Introducing the change of variables (ξ, η) in (2.14) we get

$$(2.15) \qquad \begin{aligned} &\int_{\Sigma'} a' \left(\frac{\partial u'}{\partial \xi} \frac{\partial v'}{\partial \xi} | \ \underset{\sim}{\text{grad}} \ \ \xi|^2 + \frac{\partial u'}{\partial \eta} \frac{\partial v'}{\partial \eta} | \ \underset{\sim}{\text{grad}} \ \ \eta|^2 \right) \frac{\partial(x,y)}{\partial(\xi,\eta)} d\xi d\eta \\ &= \int_{\Sigma'} f'v' \frac{\partial(x,y)}{\partial(\xi,\eta)} d\xi d\eta \quad \forall v' \in H_0^1(\Sigma'). \end{aligned}$$

Now we introduce a second change of variables,

$$(2.16) \qquad \tilde{\xi} = \int_{\xi_\Sigma^1}^{\xi} \frac{dt}{a'(t)}, \qquad \tilde{\eta} = \eta,$$

where ξ_Σ^1 is the ξ-coordinate of the left edge of Σ'. We use the notation

$$\tilde{w}'(\tilde{\xi}(\xi,\eta), \tilde{\eta}(\xi,\eta)) = w'(\xi,\eta), \quad (\xi,\eta) \in \Sigma', \quad \text{for any function } w' \text{ on } \Sigma'$$

and

$$\tilde{\Sigma}' = \text{ image of } \Sigma' \text{ under the mapping } (\tilde{\xi}, \tilde{n}).$$

Applying this change of variables to (2.15) we get

$$(2.17) \int_{\tilde{\Sigma}'} \left(\tilde{a}_1' \frac{\partial \tilde{u}'}{\partial \tilde{\xi}} \frac{\partial \tilde{v}'}{\partial \tilde{\xi}} + \tilde{a}'^2 \tilde{a}_2' \frac{\partial \tilde{u}'}{\partial \tilde{\eta}} \frac{\partial \tilde{v}'}{\partial \tilde{\eta}} \right) d\tilde{\xi} d\tilde{\eta} = \int_{\tilde{\Sigma}'} \tilde{f}' \tilde{v}' \tilde{a}' \frac{\partial(\widetilde{x,y})}{\partial(\xi,\eta)} d\tilde{\xi} d\tilde{\eta} \quad \forall \tilde{v}' \in H_0^1(\tilde{\Sigma}'),$$

where

$$(2.18) \qquad a_1'(\xi,\eta) = | \ \underset{\sim}{\text{grad}} \ \ \xi|^2 \frac{\partial(x,y)}{\partial(\xi,\eta)}, \qquad a_2'(\xi,\eta) = | \ \underset{\sim}{\text{grad}} \ \ \eta|^2 \frac{\partial(x,y)}{\partial(\xi,\eta)}.$$

To apply Theorem 2.1, we need to introduce Dirichlet boundary conditions. From condition (vi) we know that any edge of Σ is either an interior edge or a boundary edge. Then, through the correspondence determined by the mappings (ξ,η) and $(\tilde{\xi},\tilde{\eta})$, we refer to the interior and boundary edges of Σ' and $\tilde{\Sigma}'$. Now let $\tilde{\phi}' \in C^\infty(\overline{\tilde{\Sigma}'})$ with $\tilde{\phi}'(\tilde{\xi},\tilde{\eta}) = 0$ for $(\tilde{\xi},\tilde{\eta})$ near the interior edges of $\tilde{\Sigma}'$. Then for $\tilde{v}' \in H_0^1(\tilde{\Sigma}'), \tilde{\phi}'\tilde{v}' \in H_0^1(\tilde{\Sigma}')$ and we can replace $\tilde{v}'$ by $\tilde{\phi}'\tilde{v}'$ in (2.17) to get

$$\begin{aligned} &\int_{\tilde{\Sigma}} \left(\tilde{a}_1' \frac{\partial \tilde{u}'}{\partial \tilde{\xi}} \left(\tilde{\phi}' \frac{\partial \tilde{v}'}{\partial \tilde{\xi}} + \tilde{v}' \frac{\partial \tilde{\phi}'}{\partial \tilde{\xi}} \right) + \tilde{a}'^2 \tilde{a}_2' \frac{\partial \tilde{u}'}{\partial \tilde{\eta}} \left(\tilde{\phi}' \frac{\partial \tilde{v}'}{\partial \tilde{\eta}} + \tilde{v}' \frac{\partial \tilde{\phi}'}{\partial \tilde{\eta}} \right) \right) d\tilde{\xi} d\tilde{\eta} \\ &= \int_{\tilde{\Sigma}'} \tilde{f}' \tilde{v}' \tilde{\phi}' \tilde{a}' \frac{\partial(\widetilde{x,y})}{\partial(\xi,\eta)} d\tilde{\xi} d\tilde{\eta} \end{aligned}$$

or

$$\int_{\tilde\Sigma'}\left(\tilde a_1'\frac{\partial}{\partial\tilde\xi}(\tilde\phi'\tilde u')\frac{\partial\tilde v'}{\partial\tilde\xi}+\tilde a'^2\tilde a_2'\frac{\partial}{\partial\tilde\eta}(\tilde\phi'\tilde u')\frac{\partial\tilde v'}{\partial\tilde\eta}\right)d\tilde\xi d\tilde\eta=\int_{\tilde\Sigma'}F\tilde v'd\tilde\xi d\tilde\eta, \tag{2.19}$$

where

$$F=\tilde f'\tilde\phi'\tilde a'\frac{\partial(\tilde{x,y})}{\partial(\xi,\eta)}-2\tilde a_1'\frac{\partial\tilde u'}{\partial\tilde\xi}\frac{\partial\tilde\phi'}{\partial\tilde\xi}-2\tilde a'^2\tilde a_2'\frac{\partial\tilde u'}{\partial\tilde\eta}\frac{\partial\tilde\phi'}{\partial\tilde n}-\tilde u'\ \mathrm{div}_{\tilde\xi,\tilde\eta}\left(\tilde a_1'\frac{\partial\tilde\phi'}{\partial\tilde\xi},\tilde a'^2\tilde a_2'\frac{\partial\tilde\phi'}{\partial\tilde\eta}\right). \tag{2.20}$$

Writing $w=\tilde u'\tilde\phi'$, from (2.19) we get

$$\begin{cases} w\in H_0^1(\tilde\Sigma'),\\ \displaystyle\int_{\tilde\Sigma'}\left(\tilde a_1'\frac{\partial w}{\partial\tilde\xi}\frac{\partial\tilde v'}{\partial\tilde\xi}+\tilde a'^2\tilde a_2'\frac{\partial w}{\partial\tilde\eta}\frac{\partial\tilde v'}{\partial\tilde\eta}\right)d\tilde\xi d\tilde\eta=\int_{\tilde\Sigma'}F\tilde v'd\tilde\xi d\tilde\eta\quad\forall\tilde v'\in H_0^1(\tilde\Sigma'). \end{cases} \tag{2.21}$$

$w\in H_0^1(\tilde\Sigma')$ because $\tilde\phi'=0$ near the interior edges of $\tilde\Sigma'$. Because $\tilde u'$ is in $H^1(\tilde\Sigma')$, the functions ξ,η are C^2, and $\tilde a'=\tilde a'(\tilde\xi)$, we see that F is in $L^2(\tilde\Sigma')$. The system (2.21) is simply the variational formulation of

$$\begin{cases} -\dfrac{\partial}{\partial\tilde\xi}\left(\tilde a_1'\dfrac{\partial w}{\partial\tilde\xi}\right)-\tilde a'^2(\tilde\xi)\dfrac{\partial}{\partial\tilde\eta}\left(\tilde a_2'\dfrac{\partial w}{\partial\tilde\eta}\right)=F\quad\text{in }\tilde\Sigma',\\ w=0\quad\text{on }\partial\tilde\Sigma'. \end{cases} \tag{2.22}$$

The equation in (2.22) can be formally written as

$$-\tilde a_1'\frac{\partial^2 w}{\partial\tilde\xi^2}-\tilde a'^2\tilde a_2'\frac{\partial^2 w}{\partial\tilde\eta^2}=F+\frac{\partial\tilde a_1'}{\partial\tilde\xi}\frac{\partial w}{\partial\tilde\xi}+\tilde a'^2\frac{\partial\tilde a_2'}{\partial\tilde\eta}\frac{\partial w}{\partial\tilde\eta}\equiv G. \tag{2.23}$$

Denote by W the unique solution in $H^2(\tilde\Sigma')\cap H_0^1(\tilde\Sigma')$ of

$$-\tilde a_1'\frac{\partial^2 W}{\partial\tilde\xi^2}-\tilde a'^2\tilde a_2'\frac{\partial^2 W}{\partial\tilde\eta^2}=G, \tag{2.24}$$

which exists by Theorem 2.1 and which satisfies

$$\|W\|_{2,\tilde\Sigma'}\le C(\alpha,\beta,\xi,\eta)\|G\|_{0,\tilde\Sigma'}. \tag{2.25}$$

Now $w\in H_0^1(\tilde\Sigma')$ solves the same problem formally. We want to show that $w=W$ and hence that $w\in H^2(\tilde\Sigma')$ and satisfies (2.25). Writing (2.24) in divergence form we obtain

$$\begin{aligned} &-\frac{\partial}{\partial\tilde\xi}\left(\tilde a_1'\frac{\partial W}{\partial\tilde\xi}\right)-\tilde a'^2\frac{\partial}{\partial\tilde\eta}\left(\tilde a_2'\frac{\partial W}{\partial\tilde\eta}\right)\\ &\quad=-\tilde a_1'\frac{\partial^2 W}{\partial\tilde\xi^2}-\tilde a'^2\tilde a_2'\frac{\partial^2 W}{\partial\tilde\eta^2}-\frac{\partial\tilde a_1'}{\partial\tilde\xi}\frac{\partial W}{\partial\tilde\xi}-\tilde a'^2\frac{\partial\tilde a_2'}{\partial\tilde\eta}\frac{\partial W}{\partial\tilde\eta}\\ &\quad=F+\frac{\partial\tilde a_1'}{\partial\tilde\xi}\frac{\partial w}{\partial\tilde\xi}+\tilde a'^2\frac{\partial\tilde a_2'}{\partial\tilde\eta}\frac{\partial w}{\partial\tilde\eta}-\frac{\partial\tilde a_1'}{\partial\tilde\xi}\frac{\partial W}{\partial\tilde\xi}-\tilde a'^2\frac{\partial\tilde a_2'}{\partial\tilde\eta}\frac{\partial W}{\partial\tilde\eta}. \end{aligned} \tag{2.26}$$

Letting $U = W - w$ and using (2.22) and (2.26) we see that

$$(2.27)\quad \begin{cases} -\dfrac{\partial}{\partial\tilde{\xi}}\left(\tilde{a}_1'\dfrac{\partial U}{\partial\tilde{\xi}}\right) - \tilde{a}'^2\dfrac{\partial}{\partial\tilde{\eta}}\left(\tilde{a}_2'\dfrac{\partial U}{\partial\tilde{\eta}}\right) = -\dfrac{\partial\tilde{a}_1'}{\partial\tilde{\xi}}\dfrac{\partial U}{\partial\tilde{\xi}} - \tilde{a}'^2\dfrac{\partial\tilde{a}_2'}{\partial\tilde{\eta}}\dfrac{\partial U}{\partial\tilde{\eta}} & \text{in } \tilde{\Sigma}', \\ U = 0 \quad \text{on } \partial\tilde{\Sigma}', \end{cases}$$

where we understand the equation in the weak sense. It will suffice to show that $U = 0$.

Let $T : H^{-1}(\tilde{\Sigma}') \to H_0^1(\tilde{\Sigma}')$ be the solution operator corresponding to the problem (2.22), that is, let $TF = w$. Then from (2.27) we get

$$(2.28)\quad U = T\left(-\frac{\partial\tilde{a}_1'}{\partial\tilde{\xi}}\frac{\partial U}{\partial\tilde{\xi}} - \tilde{a}'^2\frac{\partial\tilde{a}_2'}{\partial\tilde{\eta}}\frac{\partial U}{\partial\tilde{\eta}}\right) \equiv AU.$$

Because $T : H^{-1}(\tilde{\Sigma}') \to H_0^1(\tilde{\Sigma}')$ is bounded and $H^0(\tilde{\Sigma}')$ is compactly contained in $H^{-1}(\tilde{\Sigma}')$, we see that $A : H_0^1(\tilde{\Sigma}') \to H_0^1(\tilde{\Sigma})$ is compact. Suppose now that $U \neq 0$. Then from (2.28) we see that 1 is an eigenvalue of A. Hence, 1 is an eigenvalue of A^*; let V be an associated eigenfunction.

We can choose $V' \in H^2(\tilde{\Sigma}') \cap H_0^1(\tilde{\Sigma}')$ so that $\|V - V'\|_{1,\tilde{\Sigma}'} < \|V\|_{1,\tilde{\Sigma}'}$. Then $(V, V')_{H^1(\tilde{\Sigma}')} \neq 0$ and from the Fredholm alternative we see that the problem

$$(2.29)\quad (I - A)Z = V'$$

has no solution in $H_0^1(\tilde{\Sigma}')$. Recalling the definition of A we see that equation (2.29) can be written

$$(2.30)\quad Z - T\left(-\frac{\partial\tilde{a}_1'}{\partial\tilde{\xi}}\frac{\partial Z}{\partial\tilde{\xi}} - \tilde{a}'^2\frac{\partial\tilde{a}_2'}{\partial\tilde{\eta}}\frac{\partial Z}{\partial\tilde{\eta}}\right) = V',$$

which can be formally written as

$$-\frac{\partial}{\partial\tilde{\xi}}\left(\tilde{a}_1'\frac{\partial Z}{\partial\tilde{\xi}}\right) - \tilde{a}'^2\frac{\partial}{\partial\tilde{\eta}}\left(\tilde{a}_2'\frac{\partial Z}{\partial\tilde{\eta}}\right) =$$
$$-\frac{\partial\tilde{a}_1'}{\partial\tilde{\xi}}\frac{\partial Z}{\partial\tilde{\xi}} - \tilde{a}'^2\frac{\partial\tilde{a}_2'}{\partial\tilde{\eta}}\frac{\partial Z}{\partial\tilde{\eta}} - \frac{\partial}{\partial\tilde{\xi}}\left(\tilde{a}_1'\frac{\partial V'}{\partial\tilde{\xi}}\right) - \tilde{a}'^2\frac{\partial}{\partial\tilde{\eta}}\left(\tilde{a}_2'\frac{\partial V'}{\partial\tilde{\eta}}\right)$$

or

$$-\tilde{a}_1'\frac{\partial^2 Z}{\partial\tilde{\xi}^2} - \tilde{a}'^2\tilde{a}_2'\frac{\partial^2 Z}{\partial\tilde{\eta}^2} = -\frac{\partial}{\partial\tilde{\xi}}\left(\tilde{a}_1'\frac{\partial V'}{\partial\tilde{\xi}}\right) - \tilde{a}'^2\frac{\partial}{\partial\tilde{\eta}}\left(\tilde{a}_2'\frac{\partial V'}{\partial\tilde{\eta}}\right).$$

But Theorem 2.1 shows that this equation has a solution Z in $H^2(\tilde{\Sigma}') \cap H_0^1(\tilde{\Sigma}')$. It is immediate that this Z solves (2.30) and hence solves (2.29), a contradiction. Therefore we conclude that $U = 0$. We have thus shown that $w = W$ and hence that $w \in H^2(\tilde{\Sigma}')$ and

$$(2.25')\quad \|w\|_{2,\tilde{\Sigma}'} \leq C(\alpha, \beta, \xi, \eta)\|G\|_{0,\tilde{\Sigma}'}.$$

Now the function $\tilde{\phi}'$ can be chosen as a cutoff function satisfying $\tilde{\phi}' = 1$ on $\tilde{O}'$, $|D\tilde{\phi}'| \leq C(\tilde{d}')^{-1}$, and $|D^2\tilde{\phi}'| \leq C(\tilde{d}')^{-2}$, where C is some positive constant, and

$$\tilde{d}' = \begin{cases} \text{dist }(\tilde{O}', \tilde{E}') & \text{if } \tilde{E}' \neq \phi, \\ 1 \quad \text{if } \tilde{E}' = \phi. \end{cases}$$

Then from the expression for G in (2.23) and from (2.20) and (2.18) we have

$$\|G\|_{0,\tilde{\Sigma}'} \le C(\alpha,\beta,\xi,\eta)(\tilde{d}')^{-2}\|f\|_{0,\Omega}.$$

Thus from (2.25′) we see that

$$\|\tilde{u}'\|_{2,\tilde{\mathcal{O}}'} \le \|w\|_{2,\tilde{\Sigma}'} \le C(\alpha,\beta,\xi,\eta,\tilde{d}')\|f\|_{0,\Omega}. \tag{2.31}$$

Now changing from $\tilde{\xi},\tilde{\eta}$ back to ξ,η in (2.31) we obtain

$$\left(\int_{\mathcal{O}'}\left\{a'\left|\frac{\partial}{\partial\xi}\left(a'\frac{\partial u'}{\partial\xi}\right)\right|^2 + a'\left|\frac{\partial^2 u'}{\partial\xi\partial\eta}\right|^2 + \frac{1}{a'}\left|\frac{\partial^2 u'}{\partial\eta^2}\right|^2\right\}d\xi d\eta\right)^{1/2} \le C(\alpha,\beta,\xi,\eta,\tilde{d}')\|f\|_{0,\Omega'},$$

which yields (2.13) because $C_1(\alpha,\beta,\xi,\eta)\tilde{d}' \le d \le C_2(\alpha,\beta,\xi,\eta)\tilde{d}'$. □

Remark 2.1. Equation (2.12), with (ξ,η) and Σ satisfying the conditions (i)–(vi), is basis for the precise meaning of the phrase "a locally varies sharply in at most one direction," which is fully formulated in §4.1 (see Remark 4.1).

Remark 2.2. If the mapping function $\xi + i\eta$ is analytic, then the above analysis is simplified because in this case the functions a_1 and a_2 in (2.18) are equal to 1.

Remark 2.3. We can define the local analogue of the space $H^L(\Omega)$ defined in (2.5). With Σ, (ξ,η), and $\mathcal{O}$ as in Theorem 2.3,

$$H^L(\mathcal{O}) = \{u : u' \in H^1(\mathcal{O}'), a'\frac{\partial u'}{\partial\xi}, \frac{\partial u'}{\partial\eta} \in H^1(\mathcal{O}')\}$$

with the norm

$$\|u\|^2_{L,\mathcal{O}} = \|u\|^2_{1,\mathcal{O}} + |u|^2_{L,\mathcal{O}},$$

where

$$|u|^2_{L,\mathcal{O}} = \int_{\mathcal{O}'}\left(a'\left|\frac{\partial}{\partial\xi}\left(a'\frac{\partial u'}{\partial\xi}\right)\right|^2 + a'\left|\frac{\partial^2 u'}{\partial\xi\partial\eta}\right|^2 + \frac{1}{a'}\left|\frac{\partial^2 u'}{\partial\eta^2}\right|^2\right)d\xi d\eta.$$

In terms of the seminorm $|u|_{L,\mathcal{O}}$, (2.13) can be stated as

$$|u|_{L,\mathcal{O}} \le C(\alpha,\beta,\xi,\eta,d)\|f\|_{0\Omega}.$$

Remark 2.4. Theorems 2.2 and 2.3 can be easily generalized to cover coefficients $a(x,y)$ of the form $a_1(x)a_2(y)$ and $a_1(\xi)a_2(\eta)$, respectively, and coefficients $a(x,y)$ that are rough in x but smooth in y and $a(\xi,\eta)$ that are rough in ξ but smooth in η, respectively.

3. Special methods for problems with straight line unidirectional coefficients. In this section we propose and analyze methods on the basis of special elements to solve the model problem

$$\begin{cases} Lu(x,y) \equiv -\dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial}{\partial x}u(x,y)\right) - \dfrac{\partial}{\partial y}\left(a(x)\dfrac{\partial}{\partial y}u(x,y)\right) = f(x,y) \quad \forall(x,y)\in\Omega, \\ u(x,y) = 0 \quad \forall(x,y)\in\partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega = \Omega_0 = (0,1)\times(0,1), f \in L^2(\Omega)$, and $a \in L^\infty(\Omega)$ is a function of x only and satisfies (1.2). This is problem (1.1) with $a(x,y)$ a straight line unidirectional coefficient. We present three approximation methods, prove they have the optimal rate of convergence, and discuss their merits. Our results are stated in terms of the constants α and β in (1.2).

3.1. Approximation Method I. For $0 < h \le 1$, let $\mathcal{C}_h$ be a triangulation of Ω by (closed) curvilinear triangles T of diameter $\le h$, where by a curvilinear triangle $T \subset \Omega$ we mean the preimage of an ordinary triangle $\tilde{T} \subset \tilde{\Omega}$ under the mapping (2.7). Corresponding to $\mathcal{C}_h$ we have a triangulation $\tilde{\mathcal{C}}_h$ of $\tilde{\Omega}$ by usual triangles. We assume that $\{\mathcal{C}_h\}_{0<h\le 1}$ satisfies a minimal angle condition,

$$h_{\tilde{T}}/\rho_{\tilde{T}} \le \sigma \quad \forall\, \tilde{T} \in \tilde{\mathcal{C}}_h \quad \forall\, 0 < h \le 1, \tag{3.2}$$

where for any bounded set $S \subset \mathbb{R}^2$,

$$h_S = \text{ diameter of } S \tag{3.3}$$

and

$$\rho_S = \text{ diameter of the largest disk contained in } \overline{S}. \tag{3.4}$$

With $\mathcal{C}_h$ we associate the space of approximating (or shape) functions

$$\begin{aligned} S_h = \{v \in L^2(\Omega) : v|_T \in \ &\text{span}\{1, \textstyle\int_0^x \frac{dt}{a(t)}, y\} \quad \forall\, T \in \mathcal{C}_h, \\ &v \text{ is continuous at the nodes of } \mathcal{C}_h, \\ &v = 0 \text{ at the boundary nodes}\}. \end{aligned} \tag{3.5}$$

As a consequence of our choice for the curvilinear triangles T we see that $S_h \subset H_0^1(\Omega)$, that is, S_h is conforming. This is easily seen by noting that the functions $1, \int_0^x \frac{dt}{a(t)}, y$ are transformed to $1, \tilde{x}, \tilde{y}$ by (2.7). Consequently, $\tilde{S}_h \equiv \{\tilde{v} : v \in S_h\}$ ($\tilde{v}$ is defined in (2.8)), the image of S_h under the mapping (2.7), is the usual space of continuous piecewise linear approximating functions with respect to $\tilde{\mathcal{C}}_h$, and S_h is conforming because $\tilde{S}_h$ is.

Our finite element approximation u_h to u is now defined by

$$\begin{cases} u_h \in S_h, \\ B(u_h, v) = \int_\Omega fv\, dx\, dy \quad \forall\, v \in S_h, \end{cases} \tag{3.6}$$

where B is defined in (1.3). The function u_h is just the Ritz approximation to u determined by the variational formulation (1.3), in the case (3.1), and the space S_h defined in (3.5). Because it is easily seen that $\tilde{u}_h$ is the Ritz approximation to $\tilde{u}$ determined by the variational formulation (2.9) and the space S_h, we could, or course, carry out the computation and the analysis on the transformed domain $\tilde{\Omega}$. We shall, however, study the approximation on the original domain Ω because this approach better illuminates the more general case of a curvilinear unidirectional coefficient studied in §4.

It is immediate that B is a bounded bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$. Furthermore, the stability condition (cf. [1]) holds, that is, we have the following.

THEOREM 3.1. *There exists a constant $\delta(\alpha) > 0$, independent of h such that for all $0 < h \le 1$,*

$$\inf_{\substack{u \in S_{1,h} \\ \|v\|_{1,\Omega}=1}} \ \sup_{\substack{v \in S_{2,h} \\ \|w\|_{1,\Omega}=1}} |B(v,w)| \ge \delta(\alpha). \tag{3.7}$$

Proof. Because $B(v,w)$ is symmetric it is sufficient to prove that B is coercive, that is, that

$$|B(v,v)| \geq \delta(\alpha)\|v\|_{1,\Omega}^2 \quad \forall v \in S_h, \qquad 0 < h \leq 1.$$

This is immediate.

Approximability here involves the approximation of the solution u by a linear combination of the shape functions $1, \int_0^x \frac{dt}{a(t)}, y$ in terms of which S_h is defined. Let the points $P_1, P_2, P_3 \in \Omega$ be the vertices of T and let $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3$ be the vertices of $\tilde{T}$ (cf. Fig. 3.1). Because the functions $1, \int_0^x \frac{dt}{a(t)}, y$ are transformed to $1, \tilde{x}, \tilde{y}$ by (2.7), we see that the interpolation problem: Given numbers w_1, w_2, w_3, find

$$w(x,y) = \alpha + \beta \int_0^x \frac{dt}{a(t)} + \gamma y \tag{3.8}$$

satisfying $w(P_i) = w_i, i = 1,2,3$, is uniquely solvable.

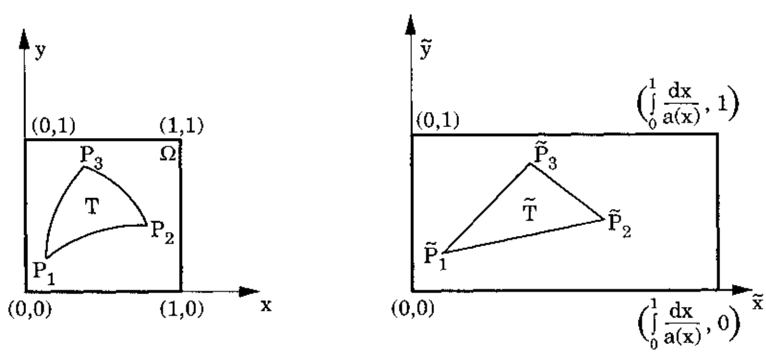


FIG. 3.1.

Suppose $u \in H^L(T)$. Then $\tilde{u} \in H^2(\tilde{T})$, and hence $\tilde{u}$ has well-defined point values for any $\tilde{P} \in \tilde{T}$. Thus u has well-defined point values for any $P \in T$, and we define the span $\{1, \int_0^x \frac{dt}{a(t)}, y\}$-interpolant of u on T by

$$d_T u = \alpha + \beta \int_0^x \frac{dt}{a(t)} + \gamma y, \qquad d_T u(P_i) = u(P_i).$$

We derive now an estimate for the difference $u - d_T u$. □

THEOREM 3.2. *There is a constant $C = C(\alpha, \beta)$, depending on α, β but independent of T and u, such that*

$$|u - d_T u|_{1,T} \leq C \frac{h_{\tilde{T}}^2}{\rho_{\tilde{T}}} |u|_{L,T} \quad \forall\, u \in H^L(T), \tag{3.9}$$

where $h_{\tilde{T}}, \rho_{\tilde{T}}$ are defined in (3.3), (3.4).

Proof. Using the transformation (2.7), we have

$$
\begin{aligned}
|u - d_T u|^2_{1,T} &= \int_T |\operatorname{grad}\ (u - d_T u)|^2 dx\, dy = \int_{\tilde{T}} \frac{1}{\tilde{a}} \left| \frac{\partial}{\partial \tilde{x}} (\tilde{u} - d_{\tilde{T}} \tilde{u}) \right|^2 d\tilde{x}\, d\tilde{y} \\
&\quad + \int_{\tilde{T}} \tilde{a} \left| \frac{\partial}{\partial \tilde{y}} (\tilde{u} - d_{\tilde{T}} \tilde{u}) \right|^2 d\tilde{x}\, d\tilde{y} \\
&\leq \max\left(\beta, \frac{1}{\alpha}\right) |\tilde{u} - d_{\tilde{T}} \tilde{u}|^2_{1,\tilde{T}},
\end{aligned}
\tag{3.10}
$$

where $d_{\tilde{T}} \tilde{u}$ is the span$\{1, \tilde{x}, \tilde{y}\}$-interpolant of $\tilde{u}$ in the triangle $\tilde{T}$. Applying the usual linear interpolation theorem (cf. [7, p. 121]), we get the bound

$$
|\tilde{u} - d_{\tilde{T}} \tilde{u}|_{1,\tilde{T}} \leq C \frac{h^2_{\tilde{T}}}{\rho_{\tilde{T}}} |\tilde{u}|_{2,\tilde{T}},
\tag{3.11}
$$

where C is an absolute constant. Inequality (3.9) is a consequence of (3.10), (3.11), and the definition of the seminorm $|\cdot|_{L,T}$, with the constant $C(\max(\beta, \frac{1}{\alpha}))^{1/2}$. □

We define now the S_h-interpolant of $u \in H^L(\Omega)$ by

$$
\begin{cases}
d_h u \in S_h, \\
d_h u(P) = u(P) \quad \text{for all nodes } P \in \mathcal{C}_h.
\end{cases}
\tag{3.12}
$$

As an easy corollary of Theorem 3.2, we can state our approximability result.

THEOREM 3.3. *There is a constant $C = C(\alpha, \beta, \sigma)$, depending on α, β, and σ but independent of u and h, such that*

$$
\|u - d_h u\|_{1,\Omega} \leq Ch |u|_{L,\Omega} \quad \forall\, u \in H^L(\Omega), \quad 0 < h \leq 1.
\tag{3.13}
$$

Proof. Because the function $u - d_h u$ is in $H^1_0(\Omega)$, from the Poincaré inequality we have

$$
\|u - d_h u\|^2_{1,\Omega} \leq C(\Omega) \sum_{T \in \mathcal{C}_h} |u - d_T u|^2_{1,T}.
\tag{3.14}
$$

Combining (3.9), (3.12), and (3.14) we get

$$
\|u - d_h u\|^2_{1,\Omega} \leq C \sum_{T \in \mathcal{C}_h} \frac{h^4_{\tilde{T}}}{\rho^2_{\tilde{T}}} |u|^2_{L,T} \leq C\sigma^2 \max_{T \in \mathcal{C}_h} h^2_{\tilde{T}} |u|^2_{L,\Omega}.
\tag{3.15}
$$

It follows immediately from the definition of the mapping (2.7) that

$$
h_{\tilde{T}} \leq \max\left(\frac{1}{\alpha}, 1\right) h_T \leq \max\left(\frac{1}{\alpha}, 1\right) h \quad \forall\, T \in \mathcal{C}_h.
\tag{3.16}
$$

Finally, estimate (3.13) follows directly from (3.15) and (3.16). □

As a consequence of the stability, approximability, and regularity results, we obtain an estimate for the error $u - u_h$ in the $H^1(\Omega)$-norm.

THEOREM 3.4. *For $f \in L^2(\Omega)$ let u be the solution to* (3.1) *and let u_h be the solution to* (3.6), *with S_h defined in* (3.5). *Then there is a constant $C = C(\alpha, \beta, \sigma)$, depending on α, β and σ but independent of f and h, such that*

$$
\|u - u_h\|_{1,\Omega} \leq Ch \|f\|_{0,\Omega}, \qquad 0 < h \leq 1.
\tag{3.17}
$$

Proof. It follows from Theorem 3.1 and standard results on the approximation of problems in variational form that

$$\|u - u_h\|_{1,\Omega} \leq C \inf_{\chi \in S_h} \|u - \chi\|_{1,\Omega}. \tag{3.18}$$

Combining (2.6), (3.13), and (3.18), and the fact that $u \in H_0^1(\Omega)$ implies $d_h u \in S_h$, we have

$$\|u - u_h\|_{1,\Omega} \leq Ch\|f\|_{0,\Omega},$$

where $C = C(\alpha, \beta, \sigma)$. □

Theorem 3.4 shows that the method defined by (3.6) is accurate and robust for the approximation of (3.1), that is, the convergence is of first order in the mesh parameter h with a constant depending on α and β, but otherwise independent of the coefficient $a(x)$. Thus the method has the same accuracy as the usual finite element method based on C^0, piecewise linear approximating functions for smooth problems.

Remark 3.1. Approximation Method I, as we have presented it, is based on a triangular mesh. One can also consider rectangular meshes. Thus for $0 < h \leq 1$, let $\mathcal{C}_h$ be a partition of Ω by rectangles R of diameter $\leq h$ and suppose $\{\mathcal{C}_h\}_{0<h\leq 1}$ satisfies a "minimal angle condition" ((diam R/diam of largest disk contained in R) $\leq \sigma$ for all $R \in \mathcal{C}_h$ and for all $0 < h \leq 1$). With $\mathcal{C}_h$ we associate the approximating functions

$$S_h = \left\{ v \in C^0(\overline{\Omega}) : v|_R \in \text{ span}\left\{1, \int_0^x \frac{dt}{a}, y, y\int_0^x \frac{dt}{a}\right\} \forall R \in \mathcal{C}_h, \right.$$
$$\left. v = 0 \quad \text{on } \partial\Omega \right\}.$$

The finite element approximation u_h is defined by (3.6) with this choice for S_h. Then it is easily seen that the arguments used to prove Theorem 3.4 yield

$$\|u - u_h\|_{1,\Omega} \leq C(\alpha, \beta, \sigma)h\|f\|_{0,\Omega}, \tag{3.17$'$}$$

the same estimate proved for triangular meshes.

Remark 3.2. Method I has an obvious one-dimensional version. This one-dimensional method differs from the standard finite element method based on C^0, piecewise linear approximating functions in that the coefficient $a(x)$ enters the finite element calculations via its element-by-element harmonic averages instead of via its averages (and the right-hand side is treated in a slightly different manner). It is referred to as a generalized displacement method (cf. [4]). In the methods presented in this paper, the coefficient $a(x, y)$ enters the calculations via various element-by-element harmonic averages and averages, that is, via various element-by-element moments of $1/a(x, y)$ and $a(x, y)$.

3.2. Approximation Method II. In Method I we chose trial or shape functions that closely approximated the unknown solution. We then used the same functions for test functions, and the stability condition was immediate. To ensure our methods were conforming, we used curvilinear triangles. In this subsection, we discuss a second method, employing the triangulation by ordinary triangles shown in Fig. 3.2, the trial functions used in Method I, and C^0 piecewise linear test functions. Now the trial space will be nonconforming, but the test space will be conforming.

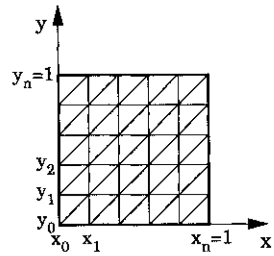


FIG. 3.2.

For $h = \frac{1}{n}, n = 2, 3, \ldots$, let $\mathcal{C}_h$ be the uniform triangulation of Ω, with nodes $(x_i, y_j) = (ih, jh), i, j = 0, \ldots, n$, shown in Fig. 3.2.

For use in our analysis, we introduce the mesh dependent spaces

$$H_h^1(\Omega) = \left\{u \in L^2(\Omega) : u|_T \in H^1(T)\ \forall\, T \in \mathcal{C}_h\right\} \tag{3.19}$$

with the norms

$$\begin{aligned} \|u\|_{1,h}^2 &= \int_\Omega u^2\, dx\, dy + |u|_{1,h}^2 \\ &= \int_\Omega u^2\, dx\, dy + \sum_{T\in\mathcal{C}_h} \int_T |\ \underset{\sim}{\text{grad}}\ \ u|^2\, dx\, dy. \end{aligned} \tag{3.20}$$

It is clear that these spaces are Hilbert spaces.

We define the bilinear form B_h on $H_h^1(\Omega) \times H_0^1(\Omega)$ by

$$B_h(u, v) = \sum_{T\in\mathcal{C}_h} \int_T a\ \ \underset{\sim}{\text{grad}}\ \ u \cdot\ \underset{\sim}{\text{grad}}\ \ v\, dx\, dy. \tag{3.21}$$

Clearly B_h is bounded on $H_h^1(\Omega) \times H_0^1(\Omega)$, with a bound that is independent of h. Moreover, $B_h(u, v) = B(u, v)$ for all $u, v \in H_0^1(\Omega)$. Now we define the trial space $S_{1,h}$ and the test space $S_{2,h}$:

$$\begin{aligned} S_{1,h} = \Big\{v \in L^2(\Omega) : v|_T \in \ \text{span}\left\{1, \int_0^x \frac{dt}{a(t)}, y\right\}\ \forall\, T \in \mathcal{C}_h, \\ v \text{ is continuous at the nodes of } \mathcal{C}_h, \\ v = 0 \text{ at the boundary nodes}\Big\} \end{aligned} \tag{3.22}$$

and

$$S_{2,h} = \left\{v \in C^0(\overline{\Omega}) : v|_T \in \ \text{span}\{1, x, y\}, v|_{\partial\Omega} = 0\right\}. \tag{3.23}$$

We remark that $S_{1,h} \not\subset H_0^1$ in general, so $S_{1,h}$ is nonconforming as mentioned above.

Our finite element approximation u_h to u is then defined by

$$\begin{cases} u_h \in S_{1,h}, \\ B_h(u_h, v) = \int_\Omega f v\, dx \quad \forall\, v \in S_{2,h}. \end{cases} \tag{3.24}$$

Note that the space $H_h^1(\Omega)$ is not well suited for a weak formulation of the exact problem (3.1). Nevertheless, the error analysis of (3.24) can be carried out in the usual way. Let us suppose that a stability condition holds for (3.24), that is, there exists $\delta = \delta(\alpha, \beta)$ such that

$$\inf_{\substack{u \in S_{1,h} \\ \|u\|_{1,h}=1}} \sup_{\substack{v \in S_{2,h} \\ \|v\|_{1,\Omega}=1}} |B_h(u,v)| \geq \delta(\alpha,\beta) > 0 \quad \forall\, 0 < h \leq 1. \tag{3.25}$$

Because $\dim S_{1,h} = \dim S_{2,h}$, (3.25) implies that (3.24) is uniquely solvable. For any $u \in H^1(\Omega)_h$, we define $P_h u$ by

$$\begin{cases} P_h u \in S_{1,h}, \\ B_h(P_h u, v) = B_h(u,v) \quad \forall\, v \in S_{2,h}. \end{cases} \tag{3.26}$$

It is clear that P_h is a projection onto $S_{1,h}$. This projection is uniformly bounded in h; in fact by (3.25) and (3.26) we have

$$\|P_h u\|_{1,h} \leq \delta(\alpha,\beta)^{-1} \sup_{\substack{v \in S_{2,h} \\ \|v\|_{1,\Omega}=1}} |B_h(P_h u, v)| \leq C(\alpha,\beta)\|u\|_{1,h} \quad \forall\, u \in H_h^1(\Omega).$$

For u the solution of (3.1) and u_h the solution of (3.24) we have for any $\chi \in S_{1,h}$,

$$\|u - u_h\|_{1,h} = \|u - P_h u\|_{1,h} = \|(u-\chi) - P_h(u-\chi)\|_{1,h} \leq [1 + C(\alpha,\beta)]\|u-\chi\|_{1,h}.$$

Thus we have proved there exists a constant $C = C(\alpha, \beta)$ such that

$$\|u - u_h\|_{1,h} \leq C \inf_{\chi \in S_{1,h}} \|u - \chi\|_{1,h} \tag{3.27}$$

(cf. [1]).

We show now that the stability condition (3.25) holds.

THEOREM 3.5. *There is a positive constant $\delta = \delta(\alpha, \beta)$ such that*

$$\inf_{\substack{u \in S_{1,h} \\ \|u\|_{1,h}=1}} \sup_{\substack{v \in S_{2,h} \\ \|v\|_{1,\Omega}=1}} |B_h(u,v)| \geq \delta(\alpha,\beta) \quad \forall\, 0 < h \leq 1. \tag{3.28}$$

Proof. Let $a_h : (0,1) \to \mathbb{R}$ denote the piecewise harmonic average of $a(x)$, that is, let

$$a_h|_{I_i} = \left\{ h^1 \int_{x_{i-1}}^{x} \frac{dt}{a(t)} \right\}^{-1}, \tag{3.29}$$

where $I_i = (x_{i-1}, x_i)$. For any $u \in S_{i,h}$, let $v \in S_{2,h}$ be defined by

$$v(P) = u(P) \quad \forall \text{ nodes } P \text{ of } \mathcal{C}_h.$$

We will verify now the relations:

$$a\frac{\partial u}{\partial x} = a_h \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}. \tag{3.30}$$

Let us first consider a triangle T of the type shown in Fig. 3.3.

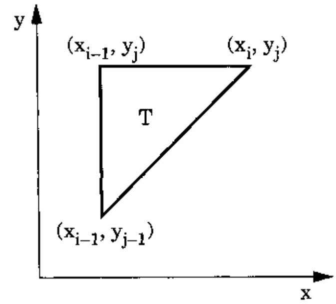


FIG. 3.3.

On T we have

$$v(x,y) = u(x_{i-1}, y_{j-1})\left\{1 - \frac{y-y_{j-1}}{h}\right\} + u(x_{i-1}, y_j)\left\{\frac{(y-y_{j-1})-(x-x_{i-1})}{h}\right\} + u(x_i, y_j)\frac{x-x_{i-1}}{h}$$

and

$$u(x,y) = u(x_{i-1}, y_{j-1})\left\{1 - \frac{y-y_{j-1}}{h}\right\} + u(x_{i-1}, y_j)\left\{\frac{(y-y_{j-1})}{h} - \frac{\int_{x_{i-1}}^{x}\frac{dt}{a(t)}}{\int_{x_{i-1}}^{x_i}\frac{dt}{a(t)}}\right\} + u(x_i, y_j)\frac{\int_{x_{i-1}}^{x}\frac{dt}{a(t)}}{\int_{x_{i-1}}^{x_i}\frac{dt}{a(t)}}.$$

From these two formulae we clearly get

$$a\frac{\partial u}{\partial x} = \frac{u(x_i,y_j)-u(x_{i-1},y_j)}{h}\frac{h}{\int_{x_{i-1}}^{x_i}\frac{dt}{a(t)}} = a_h\frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} \quad \text{in } T.$$

On the triangles of the type shown in Fig. 3.4 the proof follows the same lines. So the relations (3.30) are proved. Now using (3.30) and the Poincaré inequality we have

$$\begin{aligned} B_h(u,v) &= \sum_{T\in\mathcal{C}_h}\int_T\left\{a\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + a\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right\}dx\,dy \\ &\geq \alpha\sum_{T\in\mathcal{C}_h}\int_T|\,\underset{\sim}{\text{grad}}\quad v|^2dx\,dy = \alpha|v|_{1,\Omega}^2 \geq \frac{\alpha}{2}\|v\|_{1,\Omega}^2. \end{aligned} \tag{3.31}$$

To complete the proof we still have to bound $\|v\|_{1,\Omega}$ from below in terms of $\|u_1\|_{1,h}$. Using the relation (3.30) we obtain

$$|u|_{1,h}^2 = \sum_{T\in\mathcal{C}_h}\int_T\left\{\left(\frac{a_h}{a}\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}^2\right)\right\}dx\,dy \leq \left(\frac{\beta}{\alpha}\right)^2|v|_{1,\Omega}^2. \tag{3.32}$$

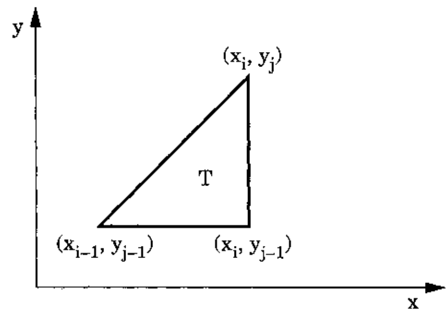


FIG. 3.4.

On the kind of triangles T shown in Fig. 3.3 we have

$$
\begin{aligned}
\int_T |u|^2 dx\,dy = \int_T |u(x_{i-1}, y_{j-1})\left\{1 - \frac{y - y_{j-1}}{h}\right\} \\
+u(x_{i-1}, y_j)\left\{\frac{y - y_{j-1}}{h} - \frac{\int_{x_{i-1}}^{x} \frac{dt}{a(t)}}{\int_{x_{i-1}}^{x_i} \frac{dt}{a(t)}}\right\} + u(x_i, y_j)\frac{\int_{x_{i-1}}^{x} \frac{dt}{a(t)}}{\int_{x_{i-1}}^{x_i} \frac{dt}{a(t)}}|^2 dx\,dy \\
\le 3\int_T \{|u(x_{i-1}, y_{j-1})|^2 + |u(x_{i-1}, y_j)|^2 + |u(x_i, y_j)|^2\} dx\,dy \\
= 3\frac{h^2}{2}\{|u(x_{i-1}, y_{j-1})|^2 + |u(x_{i-1}, y_j)|^2 + |u(x_i, y_j)|^2\} \\
\le C\int_T |v|^2 dx\,dy.
\end{aligned}
\tag{3.33}
$$

On triangles of the type shown in Fig. 3.4 we have the same estimate. Inequalities (3.32) and (3.33) show that

$$
\|u\|_{1,h} \le \{C + \left(\frac{\beta}{\alpha}\right)^2\}^{1/2}\|v\|_{1,\Omega}. \tag{3.34}
$$

From (3.31) and (3.34) it follows immediately that (3.28) holds with

$$
\delta = \frac{\alpha}{2}\{C + \left(\frac{\beta}{\alpha}\right)^2\}^{-1/2}. \quad \square
$$

For $u \in H^L(\Omega)$, let $d_h u$ be the $S_{1,h}$-interpolant of u, that is, let $d_h u$ be defined by

$$
\begin{cases} d_h u \in S_{1,h}, \\ d_h u(P) = u(P) \quad \forall \text{ nodes } P \text{ of } \mathcal{C}_h; \end{cases} \tag{3.35}
$$

$d_h u$ is well defined because u is well defined on the nodes and because the images of the vertices of any $T \in \mathcal{C}_h$ are noncolinear. In the next theorem we derive an estimate for the interpolation error $\|u - d_h u\|_{1,h}$.

THEOREM 3.6. *There is a constant $C = C(\alpha, \beta)$, depending only on α and β but independent of u and h, such that*

$$\|u - d_h u\|_{1,h} \le Ch|u|_{L,\Omega} \quad \forall\, u \in H^L(\Omega), \qquad 0 < h \le 1. \tag{3.36}$$

Proof. This proof is similar to that of Theorem 3.3. Let $u \in H^L(\Omega), T \in \mathcal{C}_h$, and R_T be the smallest rectangle containing T. Let $\tilde{T}, \tilde{R}_T$ be the images of T, R under the mapping (2.7). Then, applying the usual linear interpolation theorem as in (3.11), we have

$$|\tilde{u} - \widetilde{d_h u}|_{i,\tilde{R}_T} \le C \frac{h^2_{\tilde{R}_T}}{\rho^i_{\tilde{R}_T}} |\tilde{u}|_{2,\tilde{R}_T}, \qquad i = 0, 1. \tag{3.37}$$

To obtain the result in the original variables we note that

$$|\tilde{u} - \widetilde{d_h u}|^2_{0,\tilde{R}_T} = \int_{R_T} |u - d_h u|^2 \frac{1}{a} dx\, dy \ge \frac{1}{\beta} |u - d_h u|^2_{0,R_T} \tag{3.38}$$

and (cf. (3.10))

$$|\tilde{u} - \widetilde{d_h u}|^2_{1,R_T} \ge \min\left(\alpha, \frac{1}{\beta}\right) |u - d_h u|^2_{1,R_T}. \tag{3.39}$$

By the definition (2.5b) of the seminorm $|\cdot|_{L,R_T}$, we have

$$|\tilde{u}|_{2,\tilde{R}_T} = |u|_{L,R_T}. \tag{3.40}$$

With (3.38)–(3.40), inequality (3.37) yields

$$|u - d_h u|_{i,T} \le C(\alpha, \beta) \frac{h^2_{\tilde{R}_T}}{\rho^i_{\tilde{R}_T}} |u|_{L,R_T}, \qquad i = 0, 1. \tag{3.41}$$

From (2.7) we have

$$h_{\tilde{R}_T} \le \max\left(\frac{1}{\alpha}, 1\right) h_{R_T}, \qquad \rho_{\tilde{R}_T} \ge \min\left(1, \frac{1}{\beta}\right) \rho_{R_T}. \tag{3.42}$$

Finally, (3.36) is a consequence of (3.41) and (3.42). □

As a consequence of (3.27), (3.28), (3.36), and (2.6) we obtain the following.

THEOREM 3.7. *For $f \in L^2(\Omega)$, let u be the solution to* (3.1) *and let u_h be the solution to* (3.24). *Then there is a constant $C = C(\alpha, \beta)$ such that*

$$\|u - u_h\|_{1,h} \le Ch\|f\|_{0,\Omega}, \qquad 0 < h \le 1. \tag{3.43}$$

Remark 3.3. In Remark 3.1 we briefly outlined Method I for rectangular meshes. Here we give a rectangular mesh version of Method II. Let

$$S_{1,h} = \left\{ v \in C^0(\overline{\Omega}) : v|_R \in \text{ span}\left\{1, \int_0^x \frac{dt}{a(t)}, y, y\int_0^x \frac{dt}{a(t)}\right\} \forall\, R \in \mathcal{C}_h, \right.$$
$$\left. v = 0 \text{ on } \partial\Omega \right\}$$

and

$$S_{2,h} = \left\{ v \in C^0(\overline{\Omega}) : v|_R \in \text{ span}\{1, x, y, xy\} \,\forall\, R \in \mathcal{C}_h, v = 0 \text{ on } \partial\Omega \right\}.$$

Our finite element approximation u_h is now defined by (3.24) with this choice for $S_{1,h}$ and $S_{2,h}$. In this situation we need a hypothesis on $a(x)$ in order to ensure stability. Let

$$\phi_i(x) = \frac{\int_{x_{i-1}}^{x} \frac{dt}{a}}{\int_{x_{i-1}}^{x_i} \frac{dt}{a}} \quad \text{and} \quad \psi_i = \frac{x - x_{i-1}}{h},$$

and then let

$$A_i = h^{-1} \int_{x_{i-1}}^{x_i} a \, dx,$$
$$B_i = h^{-1} \int_{x_{i-1}}^{x_i} \phi_i a \, dx,$$
$$C_i = h^{-1} \int_{x_{i-1}}^{x_i} \psi_i \phi_i a \, dx,$$

and

$$D_i = h^{-1} \int_{x_{i-1}}^{x_i} \psi_i a \, dx.$$

We assume

$$\frac{4A_iC_i - (B_i + D_i)^2}{4C_i} \geq \gamma > 0, \quad \forall i, h.$$

Then (3.28) holds with $\delta = \delta(\alpha, \beta, \gamma) > 0$. We therefore obtain

$$\|u - u_h\|_{1,\Omega} \leq C(\alpha, \beta, \gamma) h \|f\|_{0,\Omega}. \tag{3.43$'$}$$

We remark that $S_{1,h}$ is conforming in this rectangular mesh case in contrast with the triangular mesh case in which $S_{1,h}$ is nonconforming.

3.3. Approximation Method III. In Method I we introduced curvilinear triangles to ensure the approximating functions were conforming, whereas in Method II we used a special triangulation with ordinary triangles obtaining a nonconforming method. In this section we design a conforming method based on an arbitrary triangulation with ordinary triangles.

For $0 < h \leq 1$, let $\mathcal{C}_h$ be a triangulation of Ω by ordinary triangles of diameter $\leq h$ and suppose $\{\mathcal{C}_h\}_{0<h\leq 1}$ satisfies

$$\frac{h_T}{\rho_T} \leq \sigma \quad \forall\, T \in \mathcal{C}_h, \quad \forall\, h \text{ (minimal angle condition)} \tag{3.44}$$

and

$$\frac{h}{h_T} \leq \nu \quad \forall\, T \in \mathcal{C}_h, \quad \forall\, h \text{ (quasi-uniform condition)}, \tag{3.45}$$

where h_T, ρ_T have been defined in (3.3), (3.4). Let $P_1, \dots, P_{m_h}$ be the nodes of $\mathcal{C}_h$. The function ψ_j denotes the usual piecewise linear basis function associated with the node $P_j = (x_j, y_j), j = 1, \dots, m_h$, that is, ψ_j is piecewise linear with respect to $\mathcal{C}_h$,

$$\psi_j(P_i) = \delta_{ij};$$

we remark that

$$\sum_{j=1}^{m_h} \psi_j = 1.$$

For each $j \in \{1, \dots, m_h\}$ let

$$V_j = \text{ span}\left\{\psi_j(x,y), \psi_j(x,y)\int_{x_j}^{x}\frac{dt}{a(t)}, \psi_j(x,y)(y-y_j)\right\}.$$

For the space of approximating functions we choose

$$S_h = \left\{v : \Omega \to \mathbb{R} : v = \sum_{j=1}^{m_h} v_j, v_j \in V_j, v = 0 \text{ on } \partial\Omega\right\}. \tag{3.46}$$

Our finite element approximation u_h to u is now defined by

$$\begin{cases} u_h \in S_h, \\ B(u_h, v) = \int_\Omega fv \, dx\, dy \quad \forall\, v \in S_h. \end{cases} \tag{3.47}$$

The function u_h is the Ritz approximation to u determined by the variational formulation (1.3) and the space S_h defined in (3.46). To study the convergence of the approximation (3.47), we turn our attention to an approximation result for $\{S_h\}_{0<h\leq 1}$.

First we show that we can approximate $u \in H^L(\Omega)$ by a linear combination of $1, \int_{x_j}^{x}\frac{dt}{a(t)}, y - y_j$ on S_j, where for $j = 1, \dots, m_h$, S_j is the finite element star associated with the node P_j:

$$S_j = \bigcup_{\substack{T \in \mathcal{C}_h \\ P_j \in T}} T.$$

Let R_j be the smallest rectangle with sides parallel to the axes containing S_j and let J_j be three specific vertices of R_j, including all vertices that lie on $\partial\Omega$. For any $u \in H^L(R_j)$, we define the span$\{1, \int_{x_j}^{x}\frac{dt}{a(t)}, y - y_j\}$-interpolant of u associated with P_j by

$$d_{j,h}u = \alpha + \beta\int_{x_j}^{x}\frac{dt}{a(t)} + \gamma(y - y_j), \quad (d_{j,h}u)(P) = u(P) \quad \forall\, P \in J_j. \tag{3.48}$$

We will prove the following approximability result.

THEOREM 3.8. *There is a constant $C = C(\alpha, \beta)$ such that*

$$|u - d_{j,h}u|_{i,S_j} \leq C(\alpha,\beta)\frac{h_{R_j}^2}{\rho_{R_j}^i}|u|_{L,R_j}, \quad \forall i = 0,1, \quad j = 1, \dots, m_h, \quad u \in H^L(\Omega),$$
$$0 < h \leq 1. \tag{3.49}$$

Proof. Let $j \in \{1, \dots, m_h\}$ and $u \in H^L(\Omega)$ be given. With the node P_j we associate the finite element star S_j and the rectangle R_j. $\tilde{S}_j$ and $\tilde{R}_j$ are the images under the mapping (2.7) of S_j and R_j. Clearly, $\tilde{R}_j$ is also a rectangle. It follows from (2.7) and (3.48) that $d_{\tilde{j,h}}u$ is the span$\{1, \tilde{x}, \tilde{y}\}$-interpolant of $\tilde{u}$, that is, $d_{\tilde{j,h}}u \in$ span$\{1, \tilde{x}, \tilde{y}\}$ and $d_{\tilde{j,h}}u(\tilde{P}) = \tilde{u}(\tilde{P})$ for all $P \in J_j$. Thus

$$|\tilde{u} - d_{\tilde{j,h}}u|_{i,\tilde{R}_j} \le C\frac{h^2_{\tilde{R}_j}}{\rho^1_{\tilde{R}_j}}|u|_{2,\tilde{R}_j}, \qquad i = 0, 1. \tag{3.50}$$

Returning to the original variables in (3.50) (cf. (3.38)–(3.40)), we obtain

$$|u - d_{j,h}u|_{i,R_j} \le C(\alpha, \beta)\frac{h^2_{\tilde{R}_j}}{\rho^i_{\tilde{R}_j}}|u|_{L,R_j}, \qquad i = 0, 1. \tag{3.51}$$

As in (3.42) we have

$$h_{\tilde{R}_j} \le \max\left(\frac{1}{\alpha}, 1\right) h_{R_j}, \quad \rho_{\tilde{R}_j} \ge \min\left(1, \frac{1}{\beta}\right) \rho_{R_j}. \tag{3.52}$$

Finally, (3.49) is a consequence of (3.51) and (3.52). ◻

Before stating an approximation result for $\{S_h\}_{0<h\le1}$, we prove a technical result.

LEMMA 3.1. *Let $\{\mathcal{C}_h\}_{0<h\le1}$ be a family of triangulations satisfying the minimal angle condition (3.44). Let $P_1, \dots, P_{m_h}$ denote the nodes of $\mathcal{C}_h$ and let S_j be the finite element star associated with P_j. Then we can partition the set $\{P_1, \dots, P_{m_h}\}$ of nodes into a finite number of disjoint sets $I_1, \dots, I_\ell$, with ℓ depending on σ but independent of h, such that $P_i, P_j \in I_k, i \ne j$, implies $\mathring{S}_i \cap \mathring{S}_j = \varnothing$ ($\mathring{S}_i$ denotes the interior of S_i).*

Proof. The proof is simple; in fact, we give an algorithm to construct the partition. We assimilate the triangulation to a graph, the edges being arcs. Because of the minimal angle condition, a node P_i has a limited number of arcs $\overline{P_iP_{i_k}}, Q_i = \{P_{i_k} : k = 1, \dots, \gamma_i\}$ being the neighbors of P_i, with $\gamma_i \le \gamma$, where γ depends on σ but is independent of i and h. We now state the algorithm. To construct I_1 we do the following. First take P_1 in I_1; then take the node of smallest index s in $\{P_1, \dots, P_{m_h}\}\backslash(\{P_1\} \cup Q_1)$, to ensure $\mathring{S}_1 \cap \mathring{S}_s = \varnothing$, and so on until the set $\{P_1, \dots, P_{m_h}\}\backslash(\{P_1\} \cup Q_1 \cup \{P_s\} \cup Q_s \cup \cdots)$ is empty. To construct I_2 we do the same as before with the subgraph of nodes $\{P_1, \dots, P_{m_h}\}\backslash I_1$. From the minimal angle condition and the construction of I_1, a node of this subgraph will have at most $\gamma - 1$ arcs. In this way we construct $I_3, \dots$. The algorithm will stop after at most γ steps. ◻

THEOREM 3.9. *There is a constant $C = C(\alpha, \beta, \nu, \sigma)$, depending on $\alpha, \beta, \nu, \sigma$ but independent of u and h, such that*

$$\left| u - \sum_{j=1}^{m_h} \psi_j d_{j,h}u \right|_{1,\Omega} \le Ch|u|_{L,\Omega} \quad \forall\, u \in H^L(\Omega), \qquad 0 < h \le 1. \tag{3.53}$$

Proof. Let $u \in H^L(\Omega)$ and let $I_1, \dots, I_\ell$ be the partition of the nodes of $\mathcal{C}_h$ given in Lemma 3.1. Then, because supp $\psi_j = S_j$, we have

$$
\begin{aligned}
\left|u-\sum_{j=1}^{m_h}\psi_j d_{j,h}u\right|^2_{1,\Omega} &= \left|\sum_{j=1}^{m_h}\psi_j(u-d_{j,h}u)\right|^2_{1,\Omega} \\
&= \left|\sum_{k=1}^{\ell}\sum_{j\in I_k}\psi_j(u-d_{j,h}u)\right|^2_{1,\Omega} \\
&\le \ell\sum_{k=1}^{\ell}\left|\sum_{j\in I_k}\psi_j(u-d_{j,h}u)\right|^2_{1,\Omega} \\
&= \ell\sum_{k=1}^{\ell}\int_\Omega\left|\sum_{j\in I_k}\ \underset{\sim}{\text{grad}}\ (\psi_j(u-d_{j,h}u))\right|^2 dx\,dy \\
&= \ell\sum_{k=1}^{\ell}\int_\Omega\sum_{j\in I_k}|(\ \underset{\sim}{\text{grad}}\ \ \psi_j)(u-d_{j,h}u) \\
&\qquad\qquad +\psi_j\ \underset{\sim}{\text{grad}}\ (u-d_{j,h}u)|^2 dx\,dy \\
&\le 2\ell\sum_{j=1}^{m_h}\int_{S_j}\{|(\ \underset{\sim}{\text{grad}}\ \ \psi_j)(u-d_{j,h}u)|^2 \\
&\qquad\qquad +|\psi_j\ \ \underset{\sim}{\text{grad}}\ (u-d_{j,h}u)|^2\}dx\,dy.
\end{aligned}
\tag{3.54}
$$

We note that with the assumptions (3.44), (3.45), we have the bounds

$$
|\psi_j|\le 1 \quad\text{and}\quad |\ \underset{\sim}{\text{grad}}\ \ \psi_j| \le \frac{1}{\min\limits_{P_j\in T}\rho_T} \le \frac{\sigma}{\min\limits_{P_j\in T}h_T} \le \frac{\sigma\nu}{h}.
$$

Thus from (3.54) we get

$$
\begin{aligned}
\left|u-\sum_{j=1}^{m_h}\psi_j d_{j,h}u\right|^2_{1,\Omega} &\le 2\ell\sum_{j=1}^{m_h}\left\{\frac{1}{\min\limits_{P_j\in T}\rho_T^2}\int_{S_j}|u-d_{j,h}u|^2 dx\,dy\right. \\
&\qquad \left. +\int_{S_j}|\ \underset{\sim}{\text{grad}}\ (u-d_{j,h}u)|^2 dx\,dy\right\}.
\end{aligned}
\tag{3.55}
$$

We now use Theorem 3.8 in (3.55) to get

$$
\left|u-\sum_{j=1}^{m_h}\psi_j d_{j,h}u\right|^2_{1,\Omega} \le 2\ell C(\alpha,\beta)\sum_{j=1}^{m_h}\left\{\frac{h_{R_j}^4}{\min\limits_{P_j\in T}\rho_T^2}+\frac{h_{R_j}^4}{\rho_{R_j}^2}\right\}|u|^2_{L,R_j}.
\tag{3.56}
$$

With the assumptions (3.44) and (3.45), the following estimates are obvious:

$$
h_{R_j}\le 2\max_{P_j\in T}h_T\le 2h, \rho_{R_j}\ge\min_{P_j\in T}\rho_T\ge\min_{P_j\in T}\frac{h_T}{\sigma}\ge\frac{h}{\nu\sigma}.
$$

So the inequality (3.56) becomes

$$(3.57)\qquad \left| u - \sum_{j=1}^{m_h} \psi_j d_{j,h} u \right|_{1,\Omega}^2 \leq C(\alpha, \beta, \sigma, \nu) h^2 \sum_{j=1}^{m_h} |u|_{L,R_j}^2 .$$

It remains to estimate $\sum_{j=1}^{m_h} |u|_{L,R_j}^2$. We have

$$(3.58)\qquad \sum_{j=1}^{m_h} |u|_{L,R_j}^2 \leq \sum_{T \in \mathcal{C}_h} N_T |u|_{L,T}^2,$$

where N_T = The number of rectangles R_j such that $T \cap R_j \neq \varnothing$. Let us now show that under the assumptions (3.44) and (3.45) the numbers N_T are bounded independently of h for all $T \in \mathcal{C}_h$. Let $T \in \mathcal{C}_h$ be given. If $T \cap R_j \neq \varnothing$, then P_j lies within the (closed) disk D of radius $(\sqrt{2}+1)h$ centered at the center of T. To estimate the number of nodes lying inside D, we first estimate the number $\overline{N}$ of triangles K that lie inside the disk D' of radius $(\sqrt{2}+2)h$. Because from (3.44), (3.45) we get

$$\frac{\pi h^2}{4\nu^2\sigma^2} \leq \pi \frac{h_K^2}{4\sigma^2} \leq \pi \frac{\rho_K^2}{4} \leq \text{ area}(K),$$

we have the estimate

$$\overline{N} \frac{\pi h^2}{4\nu^2\sigma^2} \leq \sum_{K \subset D'} \text{area}(K) \leq \text{ area}(D') = \pi(2+\sqrt{2})^2 h^2,$$

and hence

$$\overline{N} \leq 4\nu^2\sigma^2(2+\sqrt{2})^2.$$

So for N_T we have the bound

$$(3.59)\qquad N_T \leq 12\nu^2\sigma^2(2+\sqrt{2})^2 \quad \forall\, T \in \mathcal{C}_h, \quad 0 < h \leq 1.$$

Finally combining (3.57)–(3.59) we get

$$(3.60)\qquad \left| u - \sum_{j=1}^{m_h} \psi_j d_{j,h} u \right|_{1,\Omega}^2 \leq C h^2 |u|_{L,\Omega}^2,$$

where C depends on $\alpha, \beta, \nu, \sigma$ but not on u nor on h, which is the desired result (3.53). □

As with the Approximation Method I, the stability condition is immediate (cf. Theorem 3.1). In the same way we proved Theorem 3.4, we can prove Theorem 3.10.

THEOREM 3.10. *For $f \in L^2(\Omega)$, let u be the solution to* (3.1) *and let u_h be the solution to* (3.47). *Then there is a constant $C = C(\alpha, \beta, \nu, \sigma)$ such that*

$$(3.61)\qquad \|u - u_h\|_{1,\Omega} \leq Ch\|f\|_{0,\Omega}, \qquad 0 < h \leq 1.$$

Note that in the proof of Theorem 3.10 we use the fact that $u \in H_0^1(\Omega)$ implies $\sum_{j=1}^{m_h} \psi_j d_{j,h} u \in H_0^1(\Omega)$. This is true because J_j contains any vertices of R_j that lie on $\partial\Omega$.

3.4. Comments on Methods I, II, III. We have described three methods for approximating the solutions of problems of the type depicted on Fig. 1.1. The usual finite element method is inaccurate for these problems because the solutions may not be in $H^{1+\varepsilon}(\Omega)$ for any $\varepsilon > 0$.

Methods I and II are closely related. The central idea in these methods is to exploit the existence of a mapping from the general element to the reference element that transforms the special shape functions into polynomials and the unknown solution into a smooth function, and thereby obtain a good convergence rate. For singular corner behavior and homogeneous material, this idea is exploited in [5].

It is advantageous to use rectangular meshes in Ω that are aligned with the direction of the unidirectional composite, as described by $a(x, y)$, because they are the images of rectangular meshes on $\tilde{\Omega}$. The major difference between Methods I and II is in their treatment of the right-hand side f. Because with Method II, f enters the computation through integrals of f times the usual piecewise linear test functions (as opposed to integrals of f times the special test functions (cf. (3.5)), Method II is preferable when many right-hand sides must be treated. On the other hand, Method II is less stable than Method I, leading to larger constants in the error estimates (cf. (3.17), (3.17′), (3.43), (3.43′)). In fact, for rectangular meshes Method II may not converge for some a's; see the hypothesis on $a(x)$ in Remark 3.3. We note that for triangular meshes Method II always converges.

Method III, although similar in its use of good local approximating functions (e.g., functions satisfying the differential equation), has a rather different character than Methods I and II. In Method III the alignment of the mesh does not play a role. Finite element approximating spaces based on shape functions satisfying the differential equation have been suggested and employed in various contexts. The main problem in their use is the enforcement of some type of conformity. This can be done by various hybrid methods, for example see [2], [12]. There are, however, problems in ensuring the stability of these methods, and some of these problems have not been satisfactorily resolved. In contrast, Method III has no problems of this type and is very accurate and robust. For some computational aspects of a similar method employing harmonic polynomials in a p-version fashion and applied to the solution of Laplace's equation see [14].

4. Methods for problems with curvilinear unidirectional coefficients. The methods presented in §3 cover problems on rectangular domains with coefficients that globally vary sharply in at most one direction, that is, that are straight line unidirectional. Here we extend Methods I and III to cover coefficients that locally vary sharply in at most one direction, that is, that are curvilinear unidirectional, and to cover domains with curved boundaries. Method I′, the extension of Method I, will be based on quadrilateral and triangular elements and Method III′, the extension of Method III, will be based on triangular elements.

4.1. Method I′. Consider the boundary value problem (1.1) and suppose

- for $1 \le i \le n', (\Omega_i, \xi_i, \eta_i)$ is an open subset of Ω and a coordinate system satisfying conditions (i)–(vi) in §2 and for $n' + 1 \le i \le n$, where $n' \le n$, $(\Omega_i, \xi_i, \eta_i)$ is an open subset of Ω and a coordinate system satisfying conditions (i)–(iii), (v), (vi) in §2, that is, $\xi = \xi_i = \xi_i(x, y), \eta = \eta_i = \eta_i(x, y)$ and if (x, y) ranges over Ω_i, then (ξ_i, η_i) ranges over $\Omega_i' = (\xi^1_{\Omega_i}, \xi^2_{\Omega_i}) \times (\eta^1_{\Omega_i}, \eta^2_{\Omega_i})$, where $\Omega_i, \xi_i(x, y), \eta_i(x, y)$ satisfy conditions (i)–(iv), (vi) if $i \le n'$ and conditions (i)–(iii), (vi) if $i \ge n' + 1$ (let E_i denote the union of the interior edges of Ω_i);

- $\{\Omega_i\}_{i=1}^n$ covers Ω in the sense that

$$\Omega = \cup_{i=1}^n \Omega_i \tag{4.1}$$

and

$$\partial\Omega = \cup_{i=1}^n \{\text{int } (\overline{\Omega}_i \cap \partial\Omega) \text{ in } \partial\Omega\}; \tag{4.2}$$

- for $1 \leq i \leq n'$, we have

$$a(x, y) = a(x_i(\xi, \eta), y_i(\xi, \eta)) = a_i(\xi) \quad \forall\ (x, y) \in \Omega_i, \tag{4.3}$$

where $x = x_i(\xi, \eta), y = y_i(\xi, \eta)$ is the inverse of the mapping $\xi_i(x, y), \eta_i(x, y)$ (a_i here plays the role of a' in (2.12)), and for $n' + 1 \leq i \leq n, a(x, y)$ is smooth on Ω_i.

With $\{(\Omega_i, \xi_i, \eta_i)\}_{i=1}^n$ satisfying these assumptions, for each i let $\mathcal{O}_i$ be the result of pulling each interior edge of Ω_i (cf. condition (vi) of §2) a distance d toward the center of Ω_i. Then the $\mathcal{O}_i$'s are open sets of the type considered in Theorem 2.3 (i.e., $\mathcal{O}_i \subset \Omega_i, \mathcal{O}_i \subset\subset \Omega_i$ if $\overline{\Omega}_i \cap \partial\Omega = \varnothing$ and $\partial\mathcal{O}_i \cap \partial\Omega_i \subset \partial\Omega$ if $\overline{\Omega}_i \cap \partial\Omega \neq \varnothing$) and $\{\mathcal{O}_i\}_{i=1}^n$ satisfies (4.1)–(4.2), provided d is sufficiently small. Note that $d = \text{dist}(\mathcal{O}_i, E_i)$. We consider d to be fixed.

We note that if (1.1) corresponds to problems of the type depicted in Figs. 1.1–1.3 or to a smooth interface problem modeled as a composite material, then $\{(\Omega_i, \xi_i, \eta_i)\}_{i=1}^n$ can be chosen to satisfy the conditions outlined above.

We also note that these assumptions imply that $\partial\Omega$ is a piecewise smooth (C^2) curve with vertices with angular measure α satisfying $0 < \alpha < \pi$; in particular Ω has no reentrant vertices.

Remark 4.1. If our problem satisfies these assumptions we say that $a(x, y)$ locally varies sharply in at most one direction. Such coefficients are, as indicated earlier, also called (curvilinear) unidirectional.

With $\Omega, \Omega_1, \ldots, \Omega_n, \mathcal{O}_1, \ldots, \mathcal{O}_n, n'$, and $a(x, y)$ satisfying the hypothesis described above, we now describe the meshes we employ. For $0 < h \leq 1$, let $\mathcal{C}_h = \{T\}$ be a mesh on Ω consisting of curvilinear (closed) quadrilaterals or triangles, and satisfying the following properties:

- Each T is contained in some $\overline{\mathcal{O}}_i : T \subset \overline{\mathcal{O}}_{i(T)}, 1 \leq i(T) \leq n$;
- If $i(T) \leq n'$, then T is the image of a rectangle T' in $\Omega'_{i(T)}$ under the mapping $x = x_{i(T)} = x_{i(T)}(\xi, \eta), y = y_{i(T)} = y_{i(T)}(\xi, \eta)$, that is,

$$T = \{(x, y) : x = x_{i(T)}(\xi, \eta), y = y_{i(T)}(\xi, \eta),$$
$$\xi^1_{\Omega_{i(T)}} \leq \xi^1_T \leq \xi \leq \xi^2_T \leq \xi^2_{\Omega_{i(T)}},\ \eta^1_{\Omega_{i(T)}} \leq \eta^1_{\Omega_{i(T)}} \leq \eta^1_T \leq \eta \leq n^2_T \leq \eta^2_{\Omega_{i(T)}}\},$$

where

$$|\xi^2_T = \xi^1_T| \leq h, \qquad |\eta^2_T - \eta^1_T| \leq h, \tag{4.4a}$$

$$\sigma^{-1} \leq \frac{|\xi^2_T - \xi^1_T|}{|\eta^2_T - \eta^1_T|} \leq \sigma, \tag{4.4b}$$

where $1 \leq \sigma < \infty$ is independent of the mesh. The mapping $(\xi_{i(T)}, \eta_{i(T)})$ maps T onto $T' = (\xi^1_T, \xi^2_T) \times (\eta^1_T, \eta^2_T)$ and T' is mapped onto the reference rectangle $T^0_r = (0, 1) \times (0, 1)$ by the mapping

$$\tilde{\xi} = \tilde{\xi}_{i(T)} = \frac{\int_{\xi_T^1}^{\xi} \frac{dt}{a_{i(T)}}}{\int_{\xi_T^1}^{\xi_T^2} \frac{dt}{a_{i(T)}}} = \frac{\int_{\xi^1_{\Omega_{i(T)}}}^{\xi} \frac{dt}{a_{i(T)}} - \int_{\xi^1_{\Omega_{i(T)}}}^{\xi_T^1} \frac{dt}{a_{i(T)}}}{\int_{\xi_T^1}^{\xi_T^2} \frac{dt}{a_{i(T)}}}, \qquad \tilde{\eta} = \tilde{\eta}_{i(T)} = \frac{\eta - \eta_T^1}{\eta_T^2 - \eta_T^1}. \tag{4.5}$$

Thus the composition of these two mappings maps T onto $T^0 = T_r^0$, and the inverse, F_T, of the composition maps T^0 onto T.

- If $i(T) \geq n' + 1$, then T is the image of

$$T^0 = \begin{cases} T_r^0 & \text{if } T \text{ is a quadrilateral,} \\ T_t^0 = a \text{ reference triangle} & \text{if } T \text{ is a triangle} \end{cases}$$

under a mapping F_T satisfying F_T is invertible, and F_T and F_T^{-1} are smooth,

$$|F_T|_{1,\infty,T^0} \leq Ch, \quad |F_T|_{2,\infty,T^0} \leq Ch^2, \quad |F_T^{-1}|_{1,\infty,T} \leq Ch^{-1}, \tag{4.6}$$

and

$$|J_{F_T}|_{0,\infty,T^0} \equiv \sup_{(\tilde{\xi},\tilde{\eta}) \in T^0} |J_{F_T}(\tilde{\xi},\tilde{\eta})| \leq Ch^2, \qquad |J_{F_T^{-1}}|_{0,\infty,T} \equiv \sup_{(x,y)\in T} |J_{F_T^{-1}}(x,y)| \leq Ch^{-2}, \tag{4.7}$$

where

$$|G|_{\ell,\infty,Q} = \sup_{(t,s)\in Q} \|D^\ell G(t,s)\|_{\mathcal{L}_\ell(R^2,R^2)},$$

$$\|D^\ell G(t,s)\|_{\mathcal{L}_\ell(R^2,R^2)} = \max_{\substack{\gamma_i \in R^2 \\ \|\gamma_i\| \leq 1 \\ 1 \leq i \leq \ell}} \|D^\ell G(t,s)(\gamma_1, \ldots, \gamma_\ell)\|,$$

$$\|\cdot\| = \text{ the Euclidean vector norm on } R^2,$$

and

$$J_G(t,s) = \text{ Jacobian of } G \text{ at } (t,s).$$

The constant C in these estimates is independent of the mesh. We easily see that the mapping $F_T : T^0 \to T$ defined above for $i(T) \leq n'$ satisfies parallel assumptions. Hence we have $T = F_T(T^0)$ for all T, and it is convenient to associate the mesh $\mathcal{C} = \{T\}$ with the set of mappings $\{F_T\}$.

- The standard compatibility condition is satisfied. Suppose that T_1 and T_2 are quadrilaterals with a common edge $\ell : \ell = \overline{T}_1 \cap \overline{T}_2$. See Fig. 4.1; note that we are using two copies of the reference rectangle T^0. Assuming that ℓ is the image of the vertical line segment $\{(\tilde{\xi},\tilde{\eta}) : \tilde{\xi} = 1, 0 \leq \tilde{\eta} \leq 1\}$ under both F_{T_1} and F_{T_2}, we require that

$$F_{T_1}(1,\tilde{\eta}) = F_{T_2}(1,\tilde{\eta}), \qquad 0 \leq \tilde{\eta} \leq 1. \tag{4.8}$$

If ℓ is the image under F_{T_2} of a different edge of the reference rectangle, we would modify (4.8) in an obvious manner. Also, if either T_1 or T_2 is a triangle, the compatibility condition would be modified in an obvious way.

We point out that our mesh matches the (curved) boundary of Ω by means of blending (nonisoparametric) elements.

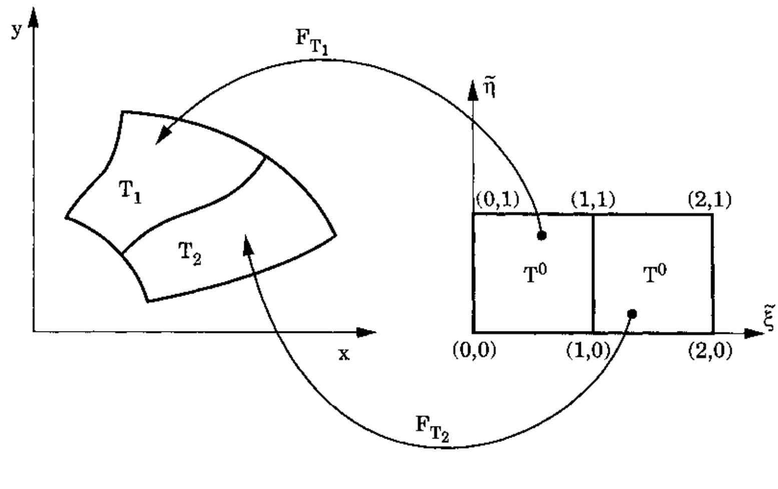


FIG. 4.1.

Remark 4.2. In the quadrilateral element case, verification of (4.6) and (4.7) usually proceeds along the following lines. Let T^* denote the straight line quadrilateral with vertices $a_i, i = 1, 2, 3, 4$, coinciding with those of T (see Fig. 4.2), let $h_T = \text{diam}\, T^*, \rho_T = $ diam of largest disk contained in T^*, and $\gamma_T = \max\{|\cos\{(a_{i+1} - a_i) \cdot (a_{i-1} - a_i)\}| 1 \leq i \leq 4(\text{mod}4)\}$ and assume

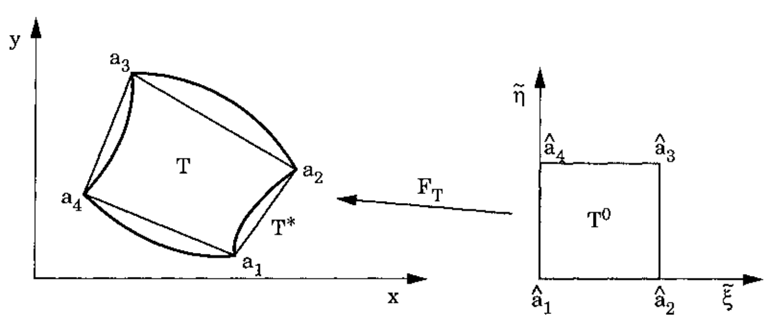


FIG. 4.2.

$$h_T \leq h, \qquad \frac{h_T}{\rho_T} \leq \sigma, \qquad \gamma_T \leq \gamma < 1,$$

where σ and γ are independent of the mesh. Let $\tilde{F}_T$ denote the bilinear mapping of T^0 onto T^* and write

$$F_T = \tilde{F}_T + \Phi.$$

One then makes assumptions on the perturbation Φ that imply (4.6) and (4.7) are satisfied. This procedure is outlined for isoparametric quadrilateral elements in [7, Exercise 4.3.9]. The parallel procedure for triangular isoparametric finite elements is carried out in [7, Thm. 4.3.3].

It remains to describe our shape functions. On $T \in \mathcal{C}_h$ we use the shape functions

$$1, F_{T,1}^{-1}(x,y), \quad F_{T,2}^{-1}(x,y) \quad F_{T,1}^{-1}(x,y), F_{T,2}^{-1}(x,y) \qquad \text{if } T \text{ is a quadrilateral} \tag{4.9}$$

and

$$1, F_{T,1}^{-1}(x,y), \quad F_{T,2}^{-1}(x,y) \qquad \text{if } T \text{ is a triangle,} \tag{4.9$'$}$$

where $F_T^{-1}(x,y) = (F_{T,1}^{-1}(x,y), F_{T,2}^{-1}(x,y))$, that is, we use the pull-back polynomials determined by the bilinear shape functions $1, \tilde{\xi}, \tilde{\eta}, \tilde{\xi}\tilde{\eta}$ in the quadrilateral case and by the linear shape functions $1, \tilde{\xi}, \tilde{\eta}$, in the triangular case. For $i(T) \leq n'$ we easily see that the functions in (4.9) are

$$1, \frac{\int_{\xi_T^1}^{\xi_{i(T)}(x,y)} \frac{dt}{a_{i(T)}}}{\int_{\xi_T^1}^{\xi_T^2} \frac{dt}{a_{i(T)}}}, \quad \frac{\eta_{i(T)}(x,y) - \eta_T^1}{\eta_T^2 - \eta_T^1}, \quad \frac{\eta_{i(T)}(x,y) - \eta_T^1}{\eta_T^2 - \eta_T^1} \frac{\int_{\xi_T^1}^{\xi_{i(T)}(x,y)} \frac{dt}{a_{i(T)}}}{\int_{\xi_T^1}^{\xi_T^2} \frac{dt}{a_{i(T)}}}. \tag{4.10}$$

Then we let

$$\begin{aligned} S_h = \{v \in L^2(\Omega) : v\big|_T \in \ &\text{span of the shape functions on } T, \\ &v \text{ is continuous at the nodes of } \mathcal{C}_h, \\ &v = 0 \text{ at the boundary nodes}\}. \end{aligned} \tag{4.11}$$

Because of the above assumptions, in particular (4.8), we see that $S_h \subset H_0^1(\Omega)$, that is, S_h is conforming. The S_h-interpolant of u is defined by

$$\begin{cases} d_h u \in S_h, \\ (d_h u)(P) = u(P) \quad \forall \text{ nodes } P \text{ of } \mathcal{C}_h. \end{cases} \tag{4.12}$$

Because of our choice of shape functions, $d_h u$ is a good approximation to u.

In Fig. 4.3 we show a typical part of the mesh on Ω. We show the sets Ω_i and $\mathcal{O}_i$ as well as the elements of the mesh. Note that both the Ω_i's and the elements fit the geometry of the fibers. In Fig. 4.4 we show the mesh in a neighborhood of the boundary of Ω. We see in particular the interior and the boundary edges of the Ω_i's. In Fig. 4.5 we show a typical mesh. We do not show the sets Ω_i and $\mathcal{O}_i$ but do show the areas where the coefficient $a(x,y)$ is smooth and where it is rough (the areas with the fibers). Note that in the area of the fibers we use quadrilaterals elements while in the area where $a(x,y)$ is smooth we use both quadrilateral and triangular elements. Obviously triangular elements cannot be avoided, but quadrilateral elements are preferable because they usually lead to higher accuracy (although with the same rate of convergence).

The approximation property of the spaces S_h is formalized in Theorem 4.1.

THEOREM 4.1. *There is a constant C depending on $\alpha, \beta, \sigma, (\xi_1, \eta_1), \ldots, (\xi_{n'}, \eta_{n'})$, $\Omega_{n'+1}, \ldots, \Omega_n, a\big|_{\Omega_{n'+1}}, \ldots, a\big|_{\Omega_n}$, the constants C in* (4.6) *and* (4.7) *and d, but independent of h and u, such that*

$$\|u - d_h u\|_{1,\Omega} \leq Ch\|f\|_{0,\Omega}, \qquad 0 < h \leq 1. \tag{4.13}$$

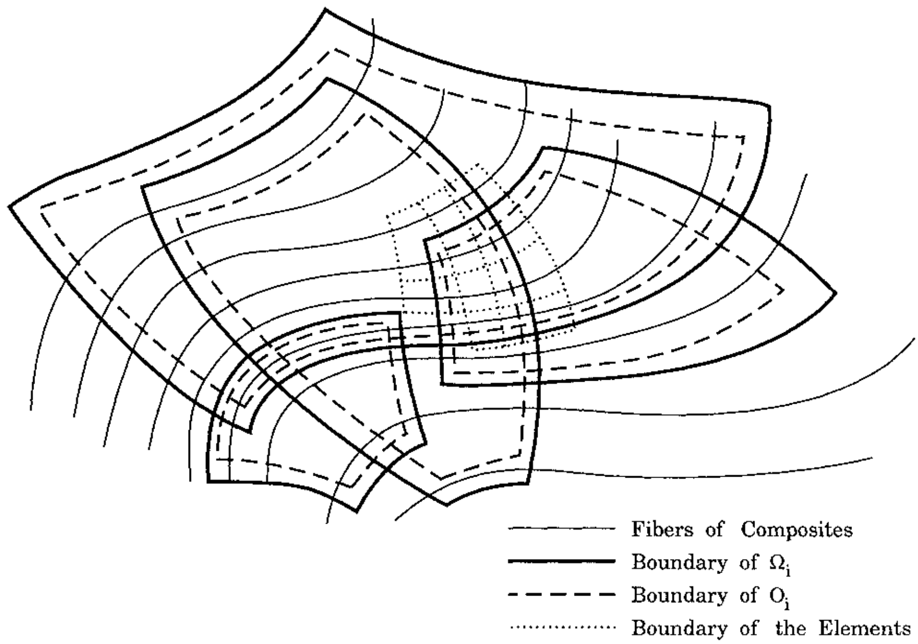


FIG. 4.3. *Typical configuration of the sets $\Omega_i, \mathcal{O}_i$, and the elements inside Ω.*

Proof. Consider $T \in \mathcal{C}_h$ and let $d_T u$ be defined by

$$d_T u \in \text{ span } \{\text{shape functions on } T\},$$

$$(d_T u)(P) = u(P), \quad \text{for all vertices } P \text{ of } T$$

(cf. (4.12)). For $i(T) \le n'$ we see that $d_T u$ is well defined by noting successively that $(d_T u)'$ (where the prime denotes the transformation from the variables (x, y) to the variables $(\xi_{i(T)}, \eta_{i(T)})$) is the

$$\text{span}\left\{1, \frac{\int_{\xi_T^1}^{\xi_{i(T)}} \frac{dt}{a_{i(T)}}}{\int_{\xi_T^1}^{\xi_T^2} \frac{dt}{a_{i(T)}}}, \frac{\eta_{i(T)} - \eta_T^1}{\eta_T^2 - \eta_T^1}, \frac{\eta_{i(T)} - \eta_T^1}{\eta_T^2 - \eta_T^1} \frac{\int_{\xi_T^1}^{\xi_{i(T)}} \frac{dt}{a_{i(T)}}}{\int_{\xi_T^1}^{\xi_T^2} \frac{dt}{a_{i(T)}}}\right\}\text{-interpolant}$$

of u' at the points P', that $(\tilde{d_T u})'$ (where the tilde denotes the transformation from the variables $(\xi_{i(T)}, \eta_{i(T)})$ to the variables

$$(4.14) \qquad \tilde{\xi}_{i(T)} = \int_{\xi^1_{\Omega_{i(T)}}}^{\xi_{i(T)}} \frac{dt}{a_{i(T)}}, \qquad \tilde{\eta}_{i(T)} = \eta_{i(T)} - \eta^1_{\Omega_{i(T)}})$$

is the span$\{1, \tilde{\xi}_{i(T)}, \tilde{\eta}_{i(T)}, \tilde{\xi}_{i(T)}\tilde{\eta}_{i(T)}\}$-interpolant of $\tilde{u}'$ at the points $\tilde{P}'$, that $\tilde{u}' \in H^2(\tilde{\mathcal{O}}'_{i(T)})$, from Theorem 2.3, and that the points $\tilde{P}'$, for P a vertex of T, form the vertices of a rectangle in $\tilde{\mathcal{O}}'_{i(T)}$. Note that the variables $\tilde{\xi}_{i(T)}$ and $\tilde{\eta}_{i(T)}$ have here

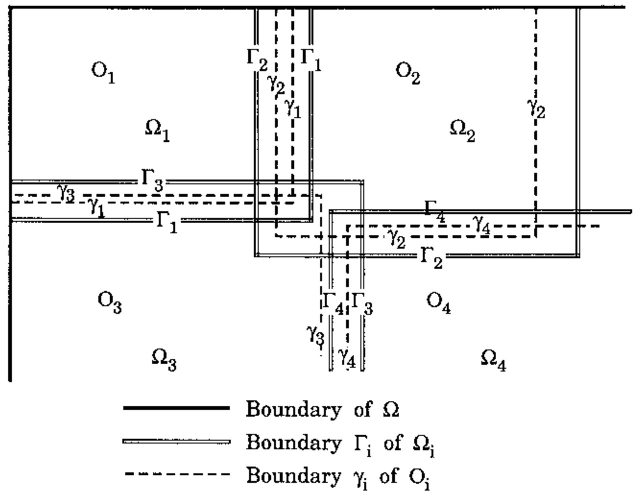


FIG. 4.4. *Typical configuration of the sets Ω_i and O_i in the neighborhood of $\partial\Omega$.*

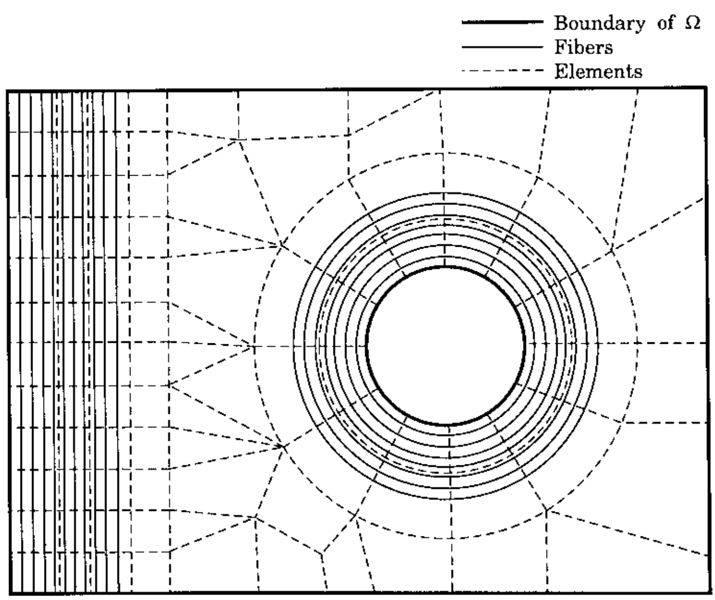


FIG. 4.5. *A complete mesh for Method I′.*

been defined differently than in (4.5). This is necessary in order that the set $\tilde{O}'_{i(T)}$ depends only on $i(T)$ and not on T. For $i(T) \geq n'+1$ we see that $d_T u$ is well defined by noting successively that $\widetilde{d_T u}$ (where the tilde denotes the transformation from the variables (x,y) to the variables $\tilde{\xi} = F^{-1}_{T,1}(x,y), \tilde{\eta} = F^{-1}_{T,2}(x,y)$) is the span$\{1,\tilde{\xi},\tilde{\eta},\tilde{\xi}\tilde{\eta}\}$-interpolant of $\tilde{u}$ if T is a quadrilateral and the span$\{1,\tilde{\xi},\tilde{\eta}\}$-interpolant of $\tilde{u}$ if T is a triangle, that $\tilde{u} \in H^2(T^0)$, from standard elliptic regularity results because $T \subset \overline{O_{i(T)}}$

and $a(x,y)$ is smooth on $\Omega_{i(T)}$, and that the points $\tilde{P}$ are the vertices of T^0. We note that these observations show that $d_h u$ in (4.12) is well defined.

Now we estimate $|u - d_T u|_{1,T}$. First suppose $i(T) \le n'$. Changing variables we obtain

$$|u - d_T u|_{1,T}^2 = \int_T \left\{ \left| \frac{\partial(u - d_T u)}{\partial x} \right|^2 + \left| \frac{\partial(u - d_T u)}{\partial y} \right|^2 \right\} dx\, dy$$

$$= \int_{T'} \left\{ \left| \frac{\partial(u' - (d_T u)')}{\partial \xi_{i(T)}} \right|^2 | \operatorname{grad} \ \xi_{i(T)}|^2 \right.$$

$$\left. + \left| \frac{\partial(u' - (d_T u)')}{\partial \eta_{i(T)}} \right|^2 | \operatorname{grad} \ \eta_{i(T)}|^2 \right\} \frac{\partial(x,y)}{\partial\left(\xi_{i(T)}, \eta_{i(T)}\right)} d\xi_{i(T)} d\eta_{i(T)}$$

$$\le C \int_{T'} \left\{ \left| \frac{\partial(u' - (d_T u)')}{\partial \xi_{i(T)}} \right|^2 + \left| \frac{\partial(u' - (d_T u)')}{\partial \eta_{i(T)}} \right|^2 \right\} d\xi_{i(T)} d\eta_{i(T)}$$

$$= C \int_{\tilde{T}'} \left\{ \frac{1}{\tilde{a}_{i(T)}} \left| \frac{\partial(\tilde{u}' - (d_{\tilde{T}} u)')}{\partial \tilde{\xi}_{i(T)}} \right|^2 + \tilde{a}_{i(T)} \left| \frac{\partial(\tilde{u}' - (d_{\tilde{T}} u)')}{\partial \tilde{\eta}_{i(T)}} \right|^2 \right\} d\tilde{\xi}_{i(T)} d\tilde{\eta}_{i(T)}$$

$$\le C(\alpha, \beta, \xi_{i(T)}, \eta_{i(T)}) |\tilde{u}' - d_{\tilde{T}} \tilde{u}'|_{1,\tilde{T}'}^2,$$

where $d_{\tilde{T}}\tilde{u}' = (d_{\tilde{T}} u)'$ denotes the span $\{1, \tilde{\xi}_{i(T)}, \tilde{\eta}_{i(T)}, \tilde{\xi}_{i(T)}\tilde{\eta}_{i(T)}\}$-interpolant of $\tilde{u}'$ on $\tilde{T}'$. Here $\tilde{\xi}_{i(T)}, \tilde{\eta}_{i(T)}$ are as defined in (4.14). Thus by standard approximation results for bilinear functions (cf. Thm. 3.1.4 in [7]) we have

$$|u - d_T u|_{1,T} \le Ch |\tilde{u}'|_{2,\tilde{T}'},$$

and hence, for $1 \le j \le n'$,

$$\text{(4.15a)} \quad \sum_{i(T)=j} |u - d_T u|_{1,T}^2 \le Ch^2 \sum_{i(T)=j} |\tilde{u}'|_{2,\tilde{T}'}^2 \le Ch^2 |\tilde{u}'|_{2,\tilde{O}_j}^2 = Ch^2 |u|_{L,O_j}^2,$$

where $C = C(\alpha, \beta, \sigma, (\xi_1, \eta_1) \dots, (\xi_{n'}, \eta_{n'}))$. Now consider $i(T) \ge n' + 1$. Using (4.6) and (4.7) and the usual proof of approximation results (cf. proof of Thm. 4.3.4 in [7]), we obtain

$$|u - d_T u|_{1,T} \le Ch(|u|_{1,T} + |u|_{2,T})$$

and hence, for $n' + 1 \le j \le n$,

$$\text{(4.15b)} \quad \sum_{i(T)=j} |u - d_T u|_{1,T}^2 \le Ch^2 \sum_{i(T)=j} \|u\|_{2,T}^2 \le Ch^2 \|u\|_{2,O_j}^2.$$

For any T, $(d_h u)\big|_T = d_T u$, and thus from (4.15a,b) we have

$$|u - d_h u|^2_{1,\Omega} = \sum_{T\in\mathcal{C}_h} |u - d_T u|^2_{1,T} = \sum_{1\le j\le n'} \sum_{i(T)=j} |u - d_T u|^2_{1,T} + \sum_{n'+1\le j\le n} \sum_{i(T)=j} |u - d_T u|^2_{1,T}$$

$$\le Ch^2 \left(\sum_{1\le j\le n'} |u|^2_{L,\mathcal{O}_j} + \sum_{n'+1\le j\le n} \|u\|^2_{2,\mathcal{O}_j} \right), \tag{4.16}$$

where $C = C(\alpha, \beta, (\xi_1, \eta_1), \dots, (\xi_{n'}, \eta_{n'})$. From Theorem 2.3 we have

$$\sum_{1\le j\le n'} |u|^2_{L,\mathcal{O}_j} \le C\|f\|^2_{0,\Omega}, \tag{4.17}$$

where $C = C(\alpha, \beta, \sigma, (\xi_1, \eta_1), \dots, (\xi_{n'}, \xi_{n'}), d)$. Because $a(x, y)$ is smooth on Ω_j for $j \ge n' + 1$, from standard elliptic regularity results we have

$$\sum_{n'+1\le j\le n} \|u\|^2_{2,\mathcal{O}_j} \le C\|f\|^2_{0,\Omega}, \tag{4.18}$$

where $C = C(\Omega_{n'+1}, \dots, \Omega_n, a\big|_{\Omega_{n'+1}}, \dots, a\big|_{\Omega_n}, d)$. As a direct consequence of (4.16)–(4.18) we get (4.13), as desired. □

Our finite element approximation u_h to u is now defined by

$$\begin{cases} u_h \in S_h, \\ B(u_h, v) = \int_\Omega f v dx \;\forall \quad v \in S_h. \end{cases} \tag{4.19}$$

Because we are using S_h for both the test and trial space, stability is immediate. Approximability has been established in Theorem 4.1. We thus have the following.

THEOREM 4.2. *Suppose* $\Omega, (\xi_1, \eta_1), \dots, (\xi_n, \eta_n), n', a(x, y)$ *satisfy the assumptions in the first part of this subsection. Suppose u is the solution of* (1.1) *and* u_h *is the solution of* (4.19). *Then there is a constant* $C = C(\alpha, \beta, \sigma, (\xi_1, \eta_1), \dots, (\xi_{n'}, \eta_{n'}), \Omega_{n'+1}, \dots, \Omega_n, a\,|_{\Omega_{n'+1}}, \dots, a\,|_{\Omega_n}, d)$ *such that*

$$\|u - u_h\|_{1,\Omega} \le Ch\|f\|_{0,\Omega} \qquad \forall\, f \in L^2(\Omega), \quad 0 < h \le 1. \tag{4.20}$$

4.2. Method III′. Consider the boundary value problem (1.1) and suppose $\{(\Omega_i, \xi_i, \eta_i)\}_{i=1}^{n'}, \{(\Omega_i, \xi_i, \eta_i)\}_{i=n'+1}^{n}, a(x, y)$, and $\{\mathcal{O}_i\}_{i=1}^{n}$ are as described in Subsection 4.1. For $0 < h \le 1$ let $\mathcal{C}_h = \{T\}$ be a triangulation of Ω by ordinary triangles together with curvilinear triangles that fit the curved part of $\partial\Omega$, all of diameter $\le h$. For any $T \in \mathcal{C}_h$ let T^* be the ordinary triangle with the same vertices as T. Then $\mathcal{C}_h^* = \{T^*\}$ is a triangulation of Ω by ordinary triangles, but

$$\bigcup_{T^*\in\mathcal{C}_h^*} T^* = \bigcup_{T\in\mathcal{C}_h} T^*$$

is a polygonal approximation to Ω and not an exact fit of Ω. We assume all T^* have diameter $\le h$ and that $\{\mathcal{C}_h^*\}_{0<h\le 1}$ satisfies the minimal angle condition (3.44) and the quasi-uniform condition (3.45). Let $\{P_j = (x_j, y_j)\}_{j=1}^{m_h}$ be the nodes of $\mathcal{C}_h^*$

and let ψ_j denote the piecewise linear basis function corresponding to P_j (and the triangulation $\mathcal{C}_h^*$). If T is curvilinear, then by restricting the domain of definition of ψ_j or by extending ψ_j as a linear function we can assume ψ_j is linear on T, and hence that ψ_j is continuous on $\overline{\Omega}$ and linear on each T. Let

$$S_j = \bigcup_{\substack{T \in \mathcal{C}_h \\ P_j \in T}} T$$

be the finite element star associated with P_j. Now it is easily seen that if h_0 is sufficiently small, then for $0 < h \le h_0$, any S_j will lie in some $\overline{\mathcal{O}}_i : S_j \subset \overline{\mathcal{O}}_{i(j)}, 1 \le i(j) \le n$. Let $S_j^{(i(j)'} \subset \overline{\mathcal{O}}'_{i(j)}$ be the image of $S_j \cap \overline{\Omega}$ under the mapping $(\xi_{i(j)}, \eta_{i(j)})$, let $R_j^{i(j)'}$ be the smallest rectangle with sides parallel to the axes containing $S_j^{i(j)'}$, and let $R_j^{i(j)}$ be the preimage of $R_j^{i(j)'}$ under $(\xi_{i(j)}, \eta_{i(j)})$. $R_j^{i(j)'} \subset \overline{\mathcal{O}}'_{i(j)}$ because $\mathcal{O}'_{i(j)}$ is a rectangle, and hence $R_j^{i(j)}$ lies in $\mathcal{O}_{i(j)}$. Define $J_j^{i(j)}$ to be three specific vertices of $R_j^{i(j)}$, including all vertices that lie on $\partial\Omega$.

Next we define our space of approximating functions. For $j = 1, \ldots, m_h$ let

$$V_j = V_j^{i(j)} = \begin{cases} \operatorname{span}\left\{\psi_j(x,y), \psi_j(x,y)\int_{\xi_{\Omega_i}^1}^{\xi_i(x,y)} \frac{dt}{a_i(t)}, \psi_j(x,y)[\eta_i(x,y) - \eta_{\Omega_i}^1]\right\} \\ \quad \text{if } 1 \le i = i(j) \le n', \\ \operatorname{span}\{\psi_j(x,y), \psi_j(x,y)(x - x_j), \psi_j(x,y)(y - y_j)\} \\ \quad \text{if } n'+1 \le i = i(j) \le n \quad \text{and} \quad \overline{\Omega}_i \bigcap \partial\Omega = \varnothing, \\ \operatorname{span}\{\psi_j(x,y), \psi_j(x,y)[\xi_i(x,y) - \xi_{\Omega_i}^1], \psi_j(x,y)[\eta_i(x,y) - \eta_{\Omega_i}^1]\} \\ \quad \text{if } n'+1 \le i = i(j) \le n \quad \text{and} \quad \overline{\Omega}_i \bigcap \partial\Omega \ne \varnothing. \end{cases} \tag{4.21}$$

The third line in this definition has been stated for the case in which the preimages of the points $(\xi_{\Omega_i}^1, \eta_{\Omega_i}^1), (\xi_{\Omega_i}^2, \eta_{\Omega_i}^1)$ lie on $\partial\Omega$. In other situations we would modify the definition in an obvious way. Then for the space of approximating functions we choose

$$S_h = \left\{ v : \Omega \to \mathbb{R} : v = \sum_{j=1}^{m_h} v_j, v_j \in V_j, v = 0 \text{ on } \partial\Omega \right\}. \tag{4.22}$$

Our finite element approximation u_h to u is now defined by

$$\begin{cases} u_h \in S_h, \\ B(u_h, v) = \int_\Omega fv\, dx\, dy \quad \forall\, v \in S_h. \end{cases} \tag{4.23}$$

Because we are using the space S_h for both the test and the trial space, stability is immediate. To study the convergence of the method (4.23) we need to prove an approximation result for the spaces $\{S_h\}_{0<h\le 1}$. This is done by combining the ideas of Subsections 3.3 and 4.1.

Let $d_{j,h}^{i(j)}u$ be defined by

$$
(4.24)\quad \begin{cases} d_{j,h}^{i(j)}u \in & \begin{cases} \operatorname{span}\left\{1, \int_{\xi_{\Omega_i}^1}^{\Omega_i(x,y)} \frac{dt}{a_i(t)}, \eta_i(x,y) - \eta_{\Omega_i}^1\right\} & \text{if } 1 \le i = i(j) \le n', \\ \operatorname{span}\{1, x - x_j, y - y_j\} & \text{if } n'+1 \le i = i(j) \le n \\ \quad \text{and } \overline{\Omega}_i \cap \partial\Omega = \varnothing, \\ \operatorname{span}\{1, \xi_i(x,y) - \xi_{\Omega_i}^1, \eta_i(x,y) - \eta_{\Omega_i}^1\} \\ \quad \text{if } n'+1 \le i = i(j) \le n \quad \text{and} \quad \overline{\Omega}_i \cap \partial\Omega \ne \varnothing. \end{cases} \\ (d_{j,h}^{i(j)}u)(P) = u(P) \quad \forall\, P \in J_j^{i(j)}. \end{cases}
$$

The function $d_{j,h}^{i(j)}u$ is a good approximation to u on S_j, as made precise in the following theorem.

THEOREM 4.3. *There is a constant $C = C(\alpha, \beta, (\xi_1, \eta_1), \ldots, (\xi_n, \eta_n))$ such that*

$$
(4.25)\quad |u - d_{j,h}^{i(j)}u|_{k,S_j} \le \begin{cases} C \frac{\left(h_{R_j^{i(j)}}\right)^2}{\left(\rho_{R_j^{i(j)}}\right)^k} u|_{L,R_j^i}(j), & \text{if } 1 \le i(j) \le n', \\ C \frac{\left(h_{R_j^{i(j)}}\right)^2}{\left(\rho_{R_j^{i(j)}}\right)^k} \|u\|_{2,R_j^i}(j), & \text{if } n'+1 \le i(j) \le n, \\ \text{for } j = 1, \ldots, m_h, k = 0, 1. \end{cases}
$$

We omit the proof of this result because it is similar to that of Theorem 3.8.

The approximability result for $\{S_h\}_{0<h\le h_0}$ is given in Theorem 4.4.

THEOREM 4.4. *There is a constant $C = C(\alpha, \beta, \nu, \sigma, (\xi_i, \eta_i), \ldots, (\xi_n, \eta_n))$ such that*

$$
(4.26)\quad \left| u - \sum_{j=1}^{m_h} \psi_j d_{j,h}^{i(j)}u \right|_{1,\Omega} \le Ch \left(\sum_{i=1}^{n'} |u|_{L,\mathcal{O}_i} + \sum_{i=n'+1}^{n} \|u\|_{2,\mathcal{O}_i} \right).
$$

We omit the proof of this result because it is similar to that of Theorem 3.9.

Finally as a consequence of (4.26), Theorem 2.3, and standard elliptic regularity results we have Theorem 4.5.

THEOREM 4.5. *Suppose $\Omega, (\xi_1, \eta_1), \ldots, (\xi_n, \eta_n), n'$, and $a(x,y)$ satisfy the assumptions in the first part of this subsection. Suppose u is the solution of (1.1) and u_h is the solution of (4.23). Then there is a constant C depending on $\alpha, \beta, \nu, \sigma, (\xi_1, \eta_1), \ldots$ $(\xi_n, \eta_n), \Omega_{n'+1}, \ldots, \Omega_n, a|_{\Omega_{n'+1}}, \ldots, a|_{\Omega_n}$, and d such that*

$$
(4.27)\quad \|u - u_h\|_{1,\Omega} \le Ch\|f\|_{0,\Omega} \quad \forall f \in L^2(\Omega), \quad 0 < h \le h_0.
$$

4.3. Comments on Method I′ and III′. The differences and similarities of Methods I′ and III′ are similar to those of Methods I and III, which are discussed in §3.4. We note that with Method I′ we have to fit the elements to the geometry of the fibers of the composite, as seen in Fig. 4.5. This is not necessary in the case of Method III′, and this freedom could be utilized in many situations. For example, suppose the coefficient is changing rapidly but not abruptly along a line. Then Method III′ could be used, leading to an enrichment of the usual finite element space by special shape functions in the neighborhood of the line.

Implementational considerations and computational studies of Methods I, II, III, I′, and III′ will be presented in a forthcoming paper.

Acknowledgment. The authors thank Professor L. C. Evans for calling the theorem of S. N. Bernstein used in §2 to their attention.

REFERENCES

[1] I. Babuška, *Error bounds for finite element method*, Numer. Math., 16 (1971), pp. 322–333.

[2] I. Babuška, J. T. Oden and J. K. Lee, *Mixed-hybrid element approximation of second-order elliptic boundary value problems, Part 2, Weak hybrid methods*, Comput. Methods Appl. Mech. Engrg., 14 (1978), pp. 1-22.

[3] I. Babuška and J. Osborn, *Numerical treatment of eigenvalue problems for differential equations with discontinuous coefficients*, Math. Comp., 32 (1978), pp. 991–1023.

[4] ———, *Generalized finite element methods: Their performance and their relation to mixed methods*, SIAM J. Numer. Anal., 20 (1983), pp. 510–536.

[5] ———, *Finite element methods for the solution of problems with rough data*, in Singularities and Constructive Methods for Their Treatment, Springer-Verlag Lecture Notes in Mathematics 1121, P. Grisvard, W. Wendland, and J. R. Whiteman, eds., Springer-Verlag, Berlin-New York, 1985, pp. 1–18.

[6] S. N. Bernstein, *Sur la généralisation du problème de Dirichlet*, Math. Ann., 62 (1906), pp. 253–271, 69 (1910), pp. 82–136.

[7] P. Ciarlet, *The Finite Element Method for Elliptic Problems*, North Holland, New York, 1978.

[8] P. G. Ciarlet, F. Natterer, and R. S. Varga, *Numerical methods of higher-order accuracy for singular nonlinear boundary value problems*, Numer. Math., 15 (1970), pp. 87–89.

[9] M. Crouzeix and J. M. Thomas, *Eléments finis et problèmes elliptiques dégénérés*, RAIRO R-3, 7 (1973), pp. 77-104.

[10] W. J. Gordon and Ch. A. Hall, *Transfinite element methods: Blending function interpolation over arbitrary curved element domains*, Numer. Math., 21 (1973), pp. 109–129.

[11] P. Grisvard, *Elliptic Problems in Non-smooth Domains*, Pitman Advanced Publishing Program, Boston-London-Melbourne, 1985.

[12] J. Jiroušek and A. Venkatesh, *Hybrid-Trefftz plane elasticity elements with p-method capabilities*, Internat. J. Numer. Methods Engrg., 35 (1992), pp. 1443–1473.

[13] O. Ladyzhenskaya and N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[14] J. M. Melenk, *Finite element methods with harmonic shape functions for solving Laplace's equation*, M.S. Thesis. Univ. of Maryland, 1992.

[15] M. Moussaoui and A. Ziani, *Sur l'approximation des solutions de certains types d'equations différentielles ou aux dérivées partielles à coéfficients non réguliers*, Annales de l'Enit, 4 (1990), pp. 9–35.

[16] J. Osborn, *The numerical solution of differential equations with rough coefficients*, in Advances in Computer Methods for Partial Differential Equations, IV, Proceedings of the Fourth IMACS International Symposium on Computer Methods for Partial Differential Equations, Lehigh University, Bethlehem, PA, (R. Vichnevetsky and R. Stepleman, eds.) 1981, pp. 9–13.

[17] ———, *Generalized finite element methods*, in Proceedings of 10th IMACS World Congress on Systems Simulations and Scientific Computation, Montreal, Canada, (R. Vichnevetsky, ed.) 1982, pp. 46–68.